DISTRIBUTIONALLY ROBUST $\mathscr{L}_1$ ADAPTIVE CONTROL FOR NONLINEAR ITÔ DIFFUSION PROCESSES

BY

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THESIS

Submitted in partial fulfillment of the requirements
for the degree of Master of Science in Mechanical Engineering
in the Graduate College of the
University of Illinois Urbana-Champaign, 2022

Urbana, Illinois

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Abstract

The problem of optimal control has been studied starting from the 17th century. Since then, people have developed different approaches for solving the optimal control problem and applied the theory of optimal control in many areas, including economics, engineering, and operations research. However, those approaches assume people have full knowledge regarding the dynamical system for the optimal control problem, which is not always possible due to the randomness and uncertainties in the real world. In recent years, data-driven approaches have been established to solve the optimal control problem with partially known dynamical systems. One of the approaches is known as reinforcement learning, which is a branch of machine learning using rewards for desired or undesired behavior. Model-based reinforcement learning, as one way for the agent to learn optimal behaviors, is more widely accepted because of higher data efficiency. To capture more information about the model during the learning process for better performance, it is also suggested the agent uses learning from distributions rather than point estimation. However, due to the uncertainties, the learned model is likely to have a distribution that is different from the true model, which can cause the agent to perform poorly in the real world and may lead to dangerous consequences.

The thesis considers the problem of errors in distributions between the learned model and the true model during the learning process from the control perspective and presents an approach to measure the difference between distributions as well as to provide a bound for the difference that can guarantee the performance for the agent. This thesis uses a continuous-time nonlinear stochastic system driven by the Wiener process. The system has the initial condition sampled from a distribution, and also has uncertainties in both the drift function part and the diffusion function part. The $\mathscr{L}_1$ adaptive controller is introduced for such a class of systems. Inside the $\mathscr{L}_1$ system, another Wiener process is introduced into both the reference system and the ideal system. Both systems have the same initial condition and corresponding initial condition distribution. The performance of $\mathscr{L}_1$ adaptive controller is then analyzed. A mean-square distance bound is provided between trajectories in actual and reference systems as well as trajectories in reference and ideal systems based on incremental Lyapunov functions. Furthermore, a bound between distributions behind trajectories between actual and ideal systems is subsequently provided. Simulation results are demonstrated how the controller can be used with traditional motion planning algorithms for obtaining safe trajectories. The code for the simulation is available at: https://github.com/SitaoZhang/StochasticMotionPlanning.jl

Acknowledgments

During my two years of master study at University of Illinois, I had a chance to meet, work and learn from many great people who provides contributions to this work in various ways. I would like to thank my adviser, Professor Naira Hovakimyan, and postdoctoral researcher, Aditya Gahlawat, for all the instructions and support during my master years. I appreciate their time and advice on all difficulties that I have encountered during the early stage for shaping my research. I would also like to thank all my friends and people for their guidance and discussions during my research. This project would not be possible without support from them. Finally, I would like to thank my parents for their support and love during this long process.

Table of contents

List of Abbreviations . . . . . **v**
List of Symbols . . . . . **vi**
Chapter 1 Introduction **1**
Chapter 2 Stochastic Dynamical Systems **5**
Chapter 3 Statistical Distance **8**
Chapter 4 ILF Based Adaptive Control **10**
Chapter 5 Analysis of the Controller Performance **17**
Chapter 6 Numerical Simulations **21**
Chapter 7 Conclusion **27**
References **28**
Appendix A Technical Results **30**
Appendix B Main Results **31**

List of Abbreviations

IAS	Incrementally Asmpytotically Stable.
IES	Incrementally Exponentially Stable.
ILF	Incremental Lyapunov Function.
SDE	Stochastic Differential Equation.
ML	Machine Learning.
RL	Reinforcement Learning.
GPR	Gaussian Processes Regression.
DNN	Deep Neural Network.
iLQG	Iterative Linear Quadratic Gaussian.

List of Symbols

$\mathbb{R}^n$	n-dimensional real coordinate space.
$L^1(\mu)$	A collection all real measurable functions f on X for which $\int_X \lvert f \rvert d\mu < \infty$ with measure μ.
$\mathcal{L}$	Infinitesimal operator.
$\underline{\sigma}_{>0}(B)$	The smallest non-zero singular value for an $n \times m$-dimension matrix B.
$C^k(U)$	The function $C : U \mapsto \mathbb{R}$ with continuous derivatives up to order k.
$C_0^k(U)$	The functions in $C^k(U)$ with compact support in U.
$\lVert \cdot \rVert$	2-norm of a vector.
$\partial_f M(x)$	The directional derivative given a matrix $M(x)$, a function $V(x)$ and a vector $f(x)$ with proper dimensions.
$L_f V$	The Lie derivative of V.
$tr(A)$	The trace of a square matrix A.
$\lVert \cdot \rVert_{\mathcal{L}_1}$	$\mathscr{L}_1$ norm.
$\mathcal{N}$	Multivariate Gaussian distribution.
W_p	The Wasserstein distance of order p.

Chapter 1

Introduction

Optimal control is an area under mathematical optimization that finds a control given a dynamical system such that an objective function related to the system is optimized. The problem of optimal control is used in many real-world situations. For example, suppose there is a car driving straight on the hilly road. The optimal control problem is then to find the optimal way for the driver to press the accelerator and shift gears so that the total travel time is minimized [1]. However, optimal control problem requires the system dynamics to be fully known, whereas, in many situations, the system dynamics is only partially known due to the environmental uncertainties [2]. In recent years, data-driven methods are developed for such optimal control problems with partially known system dynamics, such as Machine Learning (ML) [3]. The running methodology under Machine Learning is Reinforcement Learning.

These days Reinforcement Learning attracts huge interest from people in artificial intelligence and computer science community [4]. In simple words, reinforcement learning allows the agents to learn from the consequences of their own decisions instead of from the decisions of human experts and has a wide range of applications in the real world [5], [6]. There are two ways for the agent to learn the optimal behaviors to achieve the maximum reward objective with interactions from the environment. The first way is model-free RL, where the agent ignores the model and obtains the maximum reward explicitly through the trial-and-error. The other way is model-based RL, where the agent learns the model dynamics through the mapping of states and inputs, then uses the learned dynamics with a controller to decide the control and plan the optimal trajectory for the least cost in terms of the sequence of states and controls. One main advantage of model-based reinforcement learning is sample efficiency. Many models require very few samples for the agent to learn. Thus, once the cost function and the model dynamics are known or learned, then the optimal trajectory can be planned without further sampling.

For model-based reinforcement learning, one way for the agent is to learn the distribution of the model dynamics. The other way is to apply the point estimate method, where the agent learns the model containing unknowns through random samples. The main advantage of learning the distribution of the model dynamics over the point estimate method is that: learning the distribution is a more robust approach and can provide more information to the agent, such as the uncertainty or the accuracy of the estimate. However, in general for model-based reinforcement learning, one drawback is that the model itself, especially with unknowns, can contribute errors to the agent during the learning process. For an agent learning the distribution of the model dynamics, small errors will accumulate quickly during the learning process. This error accumulation may then lead the controller to plan with a suboptimal policy, which results in the distribution mismatch problem

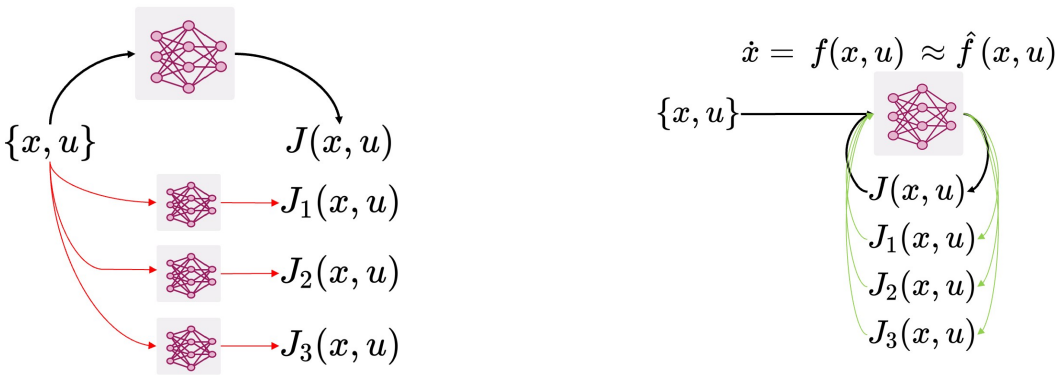

Figure 1.1: Sample graphic illustration for: (a) model-free reinforcement learning (the left graph). (b) model-based reinforcement learning (the right graph).

[7]. The distribution mismatch problem, also known as distributional shift, comes from the difference in data distribution between the learning process and the deployment process, where the agent uses the learned data to perform real-world tasks especially under the presence of noise. The distribution mismatch problem can cause the agent to perform poorly and even result in dangerous situations. Thus, we want the agent to be robust against the error from the process of learning the distribution of the model dynamics so that the error will not accumulate.

By definition, there is a strong link especially between the control theory and model-based reinforcement learning. In fact, model-based reinforcement learning uses optimal control to plan the trajectory and on the other hand, it brings optimality into the field of robust and adaptive control [8]. For example, Q-learning, which is one method of solving the reinforcement learning problems, can be viewed as the direct approach for adaptive optimal control [9]. Furthermore, robust and adaptive control are applied to handle the unknowns in the system, which is one choice to contain the effect of unknowns for error accumulation during the learning process [2].

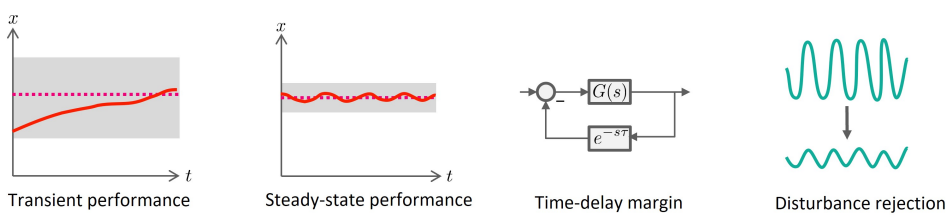

Figure 1.2: Some advantages using adaptive control: (a) Provides transient tracking performance. (b) Provides steady-state tracking performance. (c) Time-delay margin: The adaptive control can provide the tracking performance within some time delay of the control input before the system becomes unstable. (d) Disturbance rejection: The adaptive control is able to provide the stable tracking performance under the presence of disturbance in control signals.

adaptive control has no transient guarantees in general; only L1 had!! The figure is misleading. Please edit the figure and fix the caption.

For model-based reinforcement learning with optimal control problem, the stochastic model is the best choice under the presence of uncertainties or when the model dynamics is not fully known. Stochastic models include the randomness in many real-world processes by nature, which is denoted as aleatoric uncertainties [2]. Moreover, stochastic models can also represent epistemic uncertainties, which are uncertainties explainable

by the data. This can prevent the model exploitation and enable safe operation for the agent during the reinforcement learning process. Stochastic models also belong to probabilistic models. Every stochastic model has probabilities assigned to events given the model and uses the corresponding probabilities to make predictions or provide other information during the process.

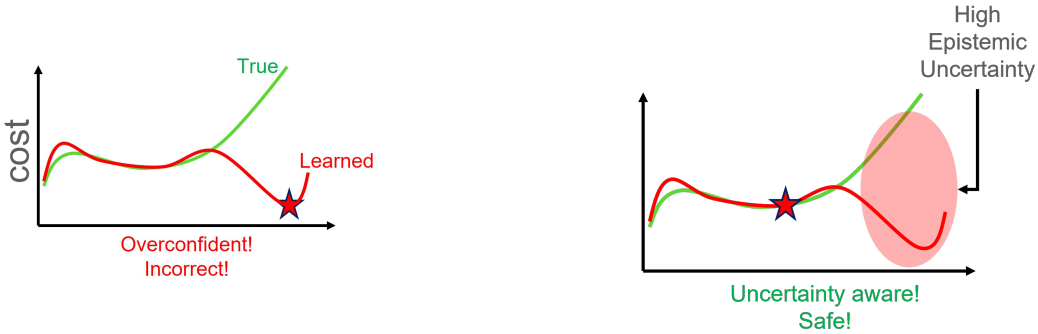

Figure 1.3: With no inclusion of epistemic uncertainty, the planned trajectory from optimal control with learned dynamics may be very different from the true trajectory as presented on the left graph, resulting in errors during the learning process. With inclusion of the epistemic uncertainty, the region with high epistemic uncertainty will be identified. Thus, the planned trajectory will avoid the high epistemic uncertainty region and ensure safety as shown on the right graph.

So for model-based reinforcement learning with stochastic models, since the learned model is stochastic, then each trajectory is different due to the randomness of the system. Thus, to prevent error accumulation during the learning process for model distribution, we want to find a way to quantify the difference between true sample trajectories and learn sample trajectories so that the agent is robust against this difference with the approach of adaptive control. However, in previous works, adaptive control is used with probability models including Gaussian Processes Regression (GPR) and ensembles of Deep Neural Networks (DNN) only to provide robustness results against parametric errors [10], [11]. Also, these probabilistic models cannot represent the true dynamics, but can only improve the predictive quantity. The problem of using adaptive control to provide robustness between state distributions in learned dynamics and in true dynamics, given the stochastic model that outputs distributions, has never been studied.

In the area of control theory, the stochastic models are written in terms of the nonlinear stochastic differential equations, which are widely used for modeling dynamical systems under the presence of noise. To study the behavior between trajectories in nonlinear stochastic dynamic systems, the analysis involving the incremental stability and nonlinear contraction theory has been investigated. Previous works include using contraction metrics to provide the mean-square distance bound between any two trajectories of a nonlinear stochastically contracting system [12], [13]. Furthermore, using the result from the mean-square distance between trajectories for the nonlinear stochastically contracting system, the extension to the convergence in the Wasserstein metric between the laws in corresponding trajectories with the given initial condition distributions is established [14], which captures the underlying geometry in the space of nonlinear stochastic systems. However, all these works have assumed that the stochastic dynamical systems are fully known. The incremental stability of nonlinear stochastic dynamic systems with unknown uncertainties has never been systematically studied. The uncertainties in the system will model both aleatoric and epistemic uncertainties for the agent in the reinforcement learning process.

In this thesis, we consider continuous-time nonlinear stochastic systems with unknown uncertainties plus

unknown parameters driven by the Wiener process, written in the form of Itô differential equation [15]. We propose the stochastic $\mathscr{L}_1$ adaptive controller to handle the uncertainties and use an incremental Lyapunov function to handle the unknown parameters and the effect of the Wiener process. Our goal is to provide a uniform performance bound for nonlinear stochastic systems with uncertainties between trajectories and between the distributions behind trajectories with the exponential decay in time and the intensity of the Wiener process.

The main contributions are: (i) We consider the nonlinear stochastic differential equations with the stochastic $\mathscr{L}_1$ adaptive controller, as opposed to considering the linear stochastic differential equations with the deterministic $\mathscr{L}_1$ adaptive controller in [16]. (ii) We extend from the deterministic nonlinear systems with nonlinear deterministic $\mathscr{L}_1$ controller [17] to stochastic nonlinear systems with the stochastic nonlinear $\mathscr{L}_1$ controller and evaluate the corresponding stochastic systems performance. (iii) Instead of using the control contraction metric [12], [13], we use a more general incremental Lyapunov function [18] to study the incremental stability of the nonlinear stochastic systems.

The outline of the thesis is organized as follows. We begin with Chapter 2 by describing the stochastic differential equation modeling the noise for our system, which will be used for our problem formulation in Chapter 4. In chapter 3, we briefly discuss the measurement for quantifying the change between two data distributions. Chapter 4 presents the problem formulation by defining our actual stochastic system with the initial condition sampled from the distribution and introduces the structure of $\mathscr{L}_1$ adaptive controller with the incorporation of incremental stability. The stability and performance of the closed-loop system is analyzed in Chapter 5. Chapter 6 provides numerical experimentation results to demonstrate our system performance. Finally, Chapter 7 concludes the paper.

Chapter 2

Stochastic Dynamical Systems

Stochastic dynamical systems are the dynamical systems that include the effect of the noise. Many processes observed in the real-world situations, such as the motion of a collection of particles, and the evolution of a stock price, contain randomness, which is also described as "noise". Stochastic dynamical systems are based on the measure theory and probability theory. Here in this chapter we cover some background for stochastic differential equations which describe the stochastic dynamical systems and cover some tools for studying continuous-time stochastic dynamical systems.

2.1 Stochastic Differential Equations

The stochastic differential equation comes from the Itô process, which is a type of stochastic process introduced by Japanese mathematician Kiyoshi Itô. The Itô process includes an integral of a process over time and another process of Brownian motion. Define a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, that models the evolution of information through time. For example, if an event, E, has occurred or not known by time t, then we have $E \in \mathcal{F}_t$. When working within the finite horizon, $[0, T]$, then we can take $\mathcal{F} = \mathcal{F}_T$. In our case, suppose all stochastic processes X_t we consider are $\mathcal{F}_t$-adapted, that is, the value of X_t is known at time t when the information represented by $\mathcal{F}_t$ is known [19]. Then the general description of the Itô process is given as:

Definition 2.1 [19] *An n-dimensional Itô process, $X_t = X(t)$, is a process that can be represented as*

$$X_t = X_0 + \int_0^t a(X_s, s)ds + \int_0^t b(X_s, s)dW \tag{2.1}$$

where W is an m-dimensional standard Brownian motion, and $a(X_t, t)$ and $b(X_t, t)$ are n-dimensional and $n \times m$-dimensional $\mathcal{F}_t$-adapted processes, respectively.

The stochastic differential equation is then the shorthand notation for Equation (2.1) such that:

Definition 2.2 [19] *The stochastic differential equation for the Itô process in Definition 2.1 is given by:*

$$dX_t = a(X_t, t)dt + b(X_t, t)dW; \qquad X_0 = x \tag{2.2}$$

where $X_0 = x$ is the starting point. As before, W is an m-dimensional standard Brownian motion, and $a(X_t, t)$ and $b(X_t, t)$ are n-dimensional and $n \times m$-dimensional $\mathcal{F}_t$-adapted processes, respectively.

Remark 2.1 [20] *$a(X_t, t)$ and $b(X_t, t)$ are also known as drift term and diffusion terms, correspondingly.*

Remark 2.2 [20] *The equivalent form of Equation* (2.2) *is:*

$$dX_t = a(X_t, t)dt + b(X_t, t)\sqrt{dt}\phi; \qquad X_0 = x$$

where $\phi \sim \mathcal{N}(0_{m\times 1}, \mathbb{I}_{m\times m})$.

2.2 Itô Lemma

A useful tool to solve SDE is Itô's Lemma. Itô's Lemma is the most important result in stochastic calculus. Here we state a general form of the result and provide some definitions derived from Itô Lemma which will be used later.

Theorem 2.1 (Itô's Lemma for multi-dimensional Itô's process) [21] *Let $X_t = X(t)$ be an n-dimensional Itô process satisfying the following SDE:*

$$dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW$$

where W is an m-dimensional standard Brownian motion, and $\mu(t, X(t))$ and $\sigma(t, X(t))$ are n-dimensional and $n \times m$-dimensional $\mathcal{F}_t$-adapted processes, respectively. If $f(t, x) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ has a continuous partial time derivative and continuous second partial space derivatives, then $F_t := f(t, X(t))$ is an n-dimensional Itô process, whose k^{th} component F_k is given by:

$$dF_k = \frac{\partial f_k}{\partial t}dt + \frac{\partial f_k}{\partial x_i}dX_i + \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}dX_i dX_j \tag{2.3}$$

summing over repeated indices, with the understanding that $dW_i dW_j = \delta_{ij}dt, dW_i dt = dt dW_i = dt dt = 0$.

Remark 2.3 *Another equivalent form for Equation* (2.3) *is:*

$$F(t) - F(0) = \int_0^t \big[\sum_i \mu_i(s, X(s))\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j} S_{i,j}(s, X(s))\frac{\partial^2 f}{\partial x_i \partial x_j} + \frac{\partial f}{\partial t}\big]ds + \int_0^t \sum_i \frac{\partial f}{\partial x_i}\sigma_i(s, X(s))dW_i$$

where $\{S_{i,j}(t, X(t))\} = S(t, X(t)) = \sigma(t, X(t))\sigma(t, X(t))^T$.
Note:

$$\begin{aligned} dX_i dX_j &= (\mu_i(t, X(t))dt + \sigma_i(t, X(t))dW_i)(\mu_j(t, X(t))dt + \sigma_j(t, X(t))dW_j) \\ &= (\sum_k \sigma_{ik}dW_k)(\sum_n \sigma_{jn}dW_n) = (\sum_k \sigma_{ik}\sigma_{jk})dt = (\sigma\sigma^T)_{i,j}dt. \end{aligned}$$

Definition 2.3 (Differential Operator) [22] *The differential operator is defined as:*

$$\mathcal{L} = \sum_i \mu_i(t, X(t))\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j} S_{i,j}(t, X(t))\frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial}{\partial t},$$

where $S_{i,j}$ is defined in Remark 2.3.

Definition 2.4 (Infinitesimal Generator) [15] *Let $\{X_t\}$ be an Itô process in $\mathbb{R}^n$. Let $\mathbf{P}^x$ denote the law of X given initial datum $X_0 = x$, and let $\mathbb{E}^x$ denote expectation with respect to $\mathbf{P}^x$. The infinitesimal generator A of X_t is defined by:*

$$Af(x) = \lim_{t \to 0} \frac{\mathbb{E}^x[f(X_t)] - f(x)}{t}; \qquad x \in \mathbb{R}^n$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a twice-differentiable function with compact support in $\mathbb{R}^n$.

Remark 2.4 *Note that A and $\mathcal{L}$ coincide on $C_0^2(\mathbb{R}^n)$.*

Chapter 3

Statistical Distance

Statistical distance measures the distance between two objects described in the area of probability theory or statistics, such as the distance between two random variables or two probability distributions. In reinforcement learning, statistical distances provide information on changes of the data related to the agent. Such changes of the data may cause performance issues. Typically, statistical distance and metrics are not the same; e.g. as Kullback–Leibler divergence. In this thesis, the Wasserstein distance will be used to measure the distance between probability distributions. As we will show later, the Wasserstein distance is also a metric that shares many useful properties.

3.1 Optimal Transport

In recent years, one of the most popular topics to learn in the area of reinforcement learning is optimal transport. The optimal transport problem is first formulated by Monge and then reproduced by Kantorovich with introduction of linear programming and duality theorem [23]. The basic idea of the optimal transport is to find a plan to transform one probability distribution into the other with the least effort. The strict mathematical formulation by Kantorovich is given as:

Definition 3.1 [24] *Let X, Y be two separable metric spaces. For any two probability measures μ, ν such that $\mu \in P(X)$ and $\nu \in P(Y)$ with a cost function $c : X \times Y \to [0, \infty)$, the optimal transport problem is formulated as:*

$$\inf\{K(\gamma) := \int_{X \times Y} c(x, y) d\gamma | \gamma \in \Pi(\mu, \nu)\} \tag{3.1}$$

where $\Pi(\mu, \nu)$ is the set of transport plans satisfying the following:

$$\forall \pi \in \Pi, \qquad \int \pi(x, y) dy = \mu(x), \qquad \int \pi(x, y) dx = \nu(y).$$

The transport plan γ which minimizes the problem in Definition 3.1 is called as the optimal transport plan between μ and ν. The minimum of the problem (3.1) can be found by solving the following dual problem:

$$\sup \big[\int \varphi(x) d\mu(x) + \int \psi(y) d\nu(y) \big],$$

among all functions $\varphi \in L^1(\mu), \psi \in L^1(\nu)$ such that:

$$\varphi(x) + \psi(y) \leq c(x, y).$$

The equivalence is known as Kantorovich duality principle and the strict proof for duality can be found in [23], [24].

3.2 Wasserstein distances

The concept of Wasserstein distance is formulated by Kantorovich during his study in optimal transport problem. The Wasserstein distance is a metric to compare two probability distributions and interpret the underlying geomeotry of the space where the Wasserstein distance is defined on. The theory of optimal transport used in this paper is borrowed from Villani [23], where a detailed exposition can be found. In this paper, the 2-Wasserstein distance is used for analysis in the following chapters. The definition for the Wasserstein space and 2-Wasserstein distance is defined as:

Definition 3.2 [23] *Let (X, d) be a Polish metric space, and let $p \in [1, \infty)$. Let μ be a probability measure on X, the Wasserstein space of order p is defined as:*

$$P_p(X) = \{\mu \in P(X); \quad \int_X d(x_0, x)^p \mu(dx) < \infty\},$$

where $x_0 \in X$ is arbitrary. This also implies that the space does not depend on the choice of the point x_0. Thus, the space $P_p(X)$ is defined such that the corresponding Wasserstein distance with any $p \in [1, \infty)$ is always finite.

Definition 3.3 [14] *For any two probability measures μ and ν on $\mathbb{R}^n$ with bounded second moments, the 2-Wassestein distance between the two measures is defined as:*

$$\begin{aligned} W_2(\mu, \nu) &= \inf(\mathbb{E}\|X - Y\|^2)^{\frac{1}{2}} \\ &= \inf_{\pi \in \Pi}[\int \int \|X - Y\|^2 d\pi(x, y)]^{\frac{1}{2}} \end{aligned}$$

where the infimum is taken over all joint probability measures in set Π on $\mathbb{R}^n \times \mathbb{R}^n$ with the marginal defined as $X \sim \mu$ and $Y \sim \nu$.

One good thing should be noted about the Wasserstein distance is that it satisfies the axiom of distance for any $p \in [1, \infty)$ with any three probability measures μ_1, μ_2 and μ_3 on X such that:

$$(nonnegativity) \quad W_p(\mu_1, \mu_2) \geq 0,$$

$$(symmetry) \quad W_p(\mu_1, \mu_2) = W_p(\mu_2, \mu_1),$$

$$(triangle\ inequality) \quad W_p(\mu_1, \mu_3) \leq W_p(\mu_1, \mu_2) + W_p(\mu_2, \mu_3)$$

Chapter 4

ILF Based Adaptive Control

In this chapter we introduce our nonlinear stochastic systems with uncertainties and the structure of proposed $\mathscr{L}_1$ controller for the uncertain nonlinear stochastic system. The key idea behind $\mathscr{L}_1$ controller is to provide estimates due to the unmatched uncertainties between the ideal system and actual system, and use the estimates in the controller to cancel the effects of uncertainties within a low-pass filter given the corresponding bandwidth. The incremental Lyapunov function is introduced to study the incremental stability for trajectories in corresponding systems.

4.1 Problem Setting

We consider the actual system described by the Itô diffusion process:

$$dx(t) = \big[f(x(t)) + B(x(t))(u(t) + h(x(t), t))\big]dt + B(x(t))(\Sigma(x(t)) + \sigma(x(t)))dw(t). \tag{4.1}$$

$$x(0) = x_0 \sim \nu_0. \tag{4.2}$$

where $x(t) \in \mathbb{R}^n$ is the actual system state; $u(t) \in \mathbb{R}^m$ is the control signal. The functions $f(x(t)) \in \mathbb{R}^n$ and $B(x(t)) \in \mathbb{R}^{n\times m}$ are known. $\Sigma(x(t)) \in \mathbb{R}^m$ is an unknown diffusion parameter describing $w(t)$, and $h(x(t), t) \in \mathbb{R}^m$ represents the uncertainties in the drift term while $\sigma(x(t)) \in \mathbb{R}^m$ represents the uncertainties in the diffusion term. $w(t)$ is a scalar Wiener process that represents the noise for the system. The initial condition x_0 is a random variable sampled from the initial condition distribution ν_0. The system without noise can be represented as [17]:

$$\begin{aligned} \dot{x}(t) &= F(x(t), u(t)) \\ &= f(x(t)) + B(x(t))(u(t) + h(x(t), t)) \end{aligned} \tag{4.3}$$

with $x(0) = x_0 \sim \nu_0$. The function F is called the drift function in Equation (4.1). Furthermore, the unperturbed/nominal dynamics ($h \equiv 0, \sigma \equiv 0$) are therefore represented as:

$$\begin{aligned} \dot{x}(t) &= \bar{F}(x(t), u(t)) \\ &= f(x(t)) + B(x(t))u(t) \end{aligned} \tag{4.4}$$

with $x(0) = x_0 \sim \nu_0$. Consider a desired/ideal control trajectory $u^*(t) \in \mathbb{R}^m$ and the induced desired/ideal state trajectory $x^*(t) \in \mathbb{R}^n$ from any planner based on unperturbed/nominal dynamics plus some Wiener process $w^*(t)$ which is independent of $w(t)$:

$$\begin{aligned} dx^*(t) &= \bar{F}(x^*(t), u^*(t))dt + B(x^*(t))\sigma(x^*(t))dw^*(t) \\ &= \big(f(x^*(t)) + B(x^*(t))u^*(t)\big)dt + B(x^*(t))\Sigma(x^*(t))dw^*(t), \end{aligned} \tag{4.5}$$

$$x^*(0) = x_0^* \sim \nu_0^*, \tag{4.6}$$

where ν_0^* and ν_0 are different initial condition distributions. Since the state $x(t)$ and $x^*(t)$ are viewed as random variables, then $x(t)$ and $x^*(t)$ have corresponding time-varying distributions, denoted as ν_t and $\nu^*(t)$. The planner ensures that the desired state-trajectory $x^*(t)$ remains in a compact safe set $\mathcal{X} \subset \mathbb{R}^n$, for all $t \geq 0$.

Remark 4.1 *For simplicity, we consider the scalar Wiener process for the actual system in Equation* (4.1)*. The results can be extended to multidimensional Wiener process cases.*

The goal is to design a control input $u(t)$ so that not only the state $x(t)$ of the true stochastic system in (1) and the state $x^*(t)$ of the desired stochastic system in (5) are bounded in the mean-square sense, but also the distribution for the state $x(t)$ of the true stochastic system in (1), ν_t, and the distribution for the state $x^*(t)$ of the desired stochastic system in (5), $\nu^*(t)$, are bounded in the Wasserstein sense, while also ensuring $x(t) \in \mathcal{X}$ for all $t \geq 0$.

The first definition we need is:

Definition 4.1 ([17], Definition 2.1) *Given a positive scalar ρ and the desired state trajectory $x^*(t)$, $\Omega(\rho, x^*(t))$ denotes the ρ-norm ball around $x^*(t)$, i.e.*

$$\Omega(\rho, x^*(t)) = \{y \in \mathbb{R}^n | \|y - x^*(t)\| \leq \rho\}.$$

The ρ-norm ball $\Omega(\rho, x^*(t))$ also induces a tube around $x^*(t)$, which is given by:

$$\mathcal{O}(\rho) = \bigcup_{t \geq 0} \Omega(\rho, x^*(t)).$$

The following assumptions will be used for further analysis:

Assumption 1. There exist constants $K_1, K_2 > 0$ such that $\forall t \in [0, T]$, $x, y \in \mathbb{R}^n$, we have

$$\begin{aligned} &\|F(x)\| + \|B(x)\Sigma(x)\| \leq K_1(1 + \|x\|), \\ &\|F(x) - F(y)\| + \|B(x)\Sigma(x) - B(y)\Sigma(y)\| \leq K_2(\|x - y\|). \end{aligned}$$

Assumption 1 is the standard rule which guarantees existence and uniqueness of solutions to be described by Itô process.

Assumption 2. ([17], Assumption 2.1) Given the positive number ρ, the ideal system trajectory satisfies $x^*(t) \in \mathcal{X}_\rho$, for all $t \geq 0$, where

$$\mathcal{X}_\rho = \mathcal{X} \ominus \mathcal{B}(\rho), \qquad \mathcal{B}(\rho) = \{y \in \mathbb{R}^n | \|y\| \leq \rho\}.$$

Assumption 3. ([17], Assumption 2.2) The desired control input $u^*(t)$ satisfies

$$\|u^*(t)\| \leq \Delta_{u^*}, \quad \forall t \geq 0,$$

with the upper bound Δ_{u^*} known from the planner, which provides the desired trajectory $x^*(t)$ in Equation (4.5).

Assumption 4. ([17], Assumption 2.3) The functions $f(x) \in \mathbb{R}^n$ and $B(x) \in \mathbb{R}^{n\times m}$ are bounded and continuously differentiable with bounded derivatives, satisfying

$$\|f(x)\| \leq \Delta_f, \quad \|\frac{\partial f(x)}{\partial x}\| \leq \Delta_{f_x}, \quad \|B(x)\| \leq \Delta_B, \quad \|\frac{\partial B(x)}{\partial x}\| \leq \Delta_{B_x},$$

where all bounds are assumed to be known.

Assumption 5. ([17], Assumption 2.4) The uncertainty $h(t,x)$ is bounded and continuously differentiable in both x and t with bounded derivatives, satisfying

$$\|h(t,x)\| \leq \Delta_h, \ \ \|\frac{\partial h(t,x)}{\partial x}\| \leq \Delta_{h_x}, \ \ \|\frac{\partial h(t,x)}{\partial t}\| \leq \Delta_{h_t}, \ \ \|\Sigma(x(t))\| \leq \Delta_{\Sigma_x}, \ \ \|\sigma(x(t))\| \leq \Delta_{\sigma_x},$$

with all bounds assumed be known for all $t \geq 0$.

Assumption 6. ([17], Assumption 2.5) The input gain matrix $B(x)$ has full column rank. Furthermore, the Moore-Penrose inverse of $B(x)$ defined as $B^\dagger(x) = (B^T(x)B(x))^{-1}B^T(x)$ satisfies the following bound:

$$\|B^\dagger(x)\| \leq \Delta_{B^\dagger}, \quad \|\frac{\partial B^\dagger(x)}{\partial x}\| \leq \Delta_{B^\dagger_x}$$

for all $x \in \mathcal{O}(\rho)$.

4.2 Incremental Stability

Consider the unperturbed/nominal system as in Equation (4.4):

$$\begin{aligned}\dot{x} &= \bar{F}(x(t), u(t)) \\ &= f(x(t)) + B(x(t))u(t).\end{aligned}$$

Recall some definitions about the stability properties for the unperturbed/nominal system.

Definition 4.2 [25] *A set $M \subset \mathbb{R}^n$ is forward invariant with respect to the system* (4.4) *if every solution of* (4.4) *starting from a point of M remains in M.*

Definition 4.3 [18] *Consider the system* (4.4) *under the control $u = \alpha(x,t)$, the solution of which is forward invariant in $\mathcal{E} \subset \mathbb{R}^n$. The closed-loop system in $\mathcal{E}$ is* (IES) *incrementally exponentially stable if $\forall (x_1, x_2) \in \mathcal{E}$*

$$\|X(t, x_1, \alpha(x_1,t)) - X(t, x_2, \alpha(x_2,t))\| \leq k_1 e^{-k_2 t}\|x_1 - x_2\|$$

holds for any $t \geq 0$ and for some constants $k_1, k_2 > 0$.

Definition 4.4 [18] *For the system $\dot{x} = \bar{F}(x,t)$, a function $V : \mathcal{E} \times \mathcal{E} \to \mathbb{R}^+$ is called IES Lyapunov function, if*

$$L_{\bar{F}(x,t)}V(x,\xi) + L_{\bar{F}(\xi,t)}V(x,\xi) \leq -2\lambda V(x,\xi) \tag{4.7}$$

for some $\lambda > 0$, and for some $\underline{\alpha}, \bar{\alpha} > 0$ satisfying

$$\underline{\alpha}\|x-\xi\|^2 \leq V(x,\xi) \leq \bar{\alpha}\|x-\xi\|^2. \tag{4.8}$$

The IES of the system $\dot{x} = \bar{F}(x,t)$ is equivalent to the existence of an IES Lyapunov function for set stability of $x = \xi$, by considering an auxiliary dynamics $\dot{\xi} = \bar{F}(\xi,t)$.

Now we place some assumptions on the nominal/unperturbed dynamics and the IES Lyapunov function $V(x,\xi)$:

Assumption 7. The nominal/unperturbed dynamics in Equation (4.4) admits an incremental exponentially stable Lyapunov function with positive numbers λ, $\bar{\alpha}$ and $\underline{\alpha}$ as in the Definition 4.4.

Assumption 8. [18] The IES Lyapunov function $V(x,\xi)$ is twice-differentiable with respect to x and ξ with bounded first and second derivatives with respect to x and ξ such that:

$$\|\frac{\partial V}{\partial x}\| \leq c_3\|x-\xi\|, \quad \|\frac{\partial V}{\partial \xi}\| \leq c_3\|x-\xi\|, \quad \|\frac{\partial^2 V}{\partial x^2}\| \leq 2c_4, \quad \|\frac{\partial^2 V}{\partial \xi^2}\| \leq 2c_4$$

for some $c_3, c_4 > 0$. Together with Assumption 4, 5, Assumption 8 will yield the following:

$$\begin{aligned}
&tr\big((\Sigma(x(t)) + \sigma(x(t)))^T B(x(t))^T \frac{\partial^2 V}{\partial x^2} B(x(t))(\Sigma(x(t)) + \sigma(x(t)))\big) \\
&\leq 2c_4 tr\big((\Sigma(x(t)) + \sigma(x(t)))^T B(x(t))^T B(x(t))(\Sigma(x(t)) + \sigma(x(t)))\big) \\
&= 2c_4\|B(x(t))\big(\Sigma(x(t)) + \sigma(x(t))\big)\|^2 \\
&\leq 2c_4 {\Delta_B}^2(\Delta_{\Sigma_x} + \Delta_{\sigma_x})^2 \\
&= 2C_1; \\
&tr\big((-\eta_r(t) + \sigma(x_r(t)) + \Sigma(x_r(t)))^T B(x_r(t))^T \frac{\partial^2 V}{\partial {x_r}^2} B(x_r(t))(-\eta_r(t) + \sigma(x(t)) + \Sigma(x(t)))\big) \\
&\leq 2c_4 tr\big((-\eta_r(t) + \sigma(x_r(t)) + \Sigma(x_r(t)))^T B(x_r(t))^T B(x_r(t))(-\eta_r(t) + \sigma(x_r(t)) + \Sigma(x_r(t)))\big) \\
&= 2c_4\|B(x_r(t))\big(-\eta_r(t) + \sigma(x_r(t)) + \Sigma(x_r(t))\big)\|^2 \\
&\leq 2c_4 {\Delta_B}^2(\Delta_{\sigma_{x_r}} + \Delta_{\Sigma_{x_r}})^2 \\
&= 2C_2; \\
&tr\big(\Sigma(x^*(t))^T B(x^*(t))^T \frac{\partial^2 V}{\partial {x^*}^2} B(x^*(t))\Sigma(x^*(t))\big) \\
&\leq 2c_4 tr\big(\Sigma(x^*(t))^T B(x^*(t))^T B(x^*(t))\Sigma(x^*(t))\big) \\
&= 2c_4\|B(x^*(t))\Sigma(x^*(t))\|^2 \\
&\leq 2c_4 {\Delta_B}^2 {\Delta_{\Sigma_x^*}}^2 \\
&= 2C_3.
\end{aligned}$$

4.3 $\mathscr{L}_1$ Adaptive Control

We now show the structure of the designed adaptive controller for the uncertain stochastic nonlinear system in Equation (4.1). Before proceeding to the detailed description about each component inside the controller, the following equations of constants are introduced for significance about the results and analysis in the following chapter of the thesis:

$$\Delta_{\delta_u} = \frac{1}{2} \sup_{x \in \mathcal{O}(\rho)} \Big(\frac{\bar{\lambda}(L^{-T}(x)F(x)L^{-1}(x))}{\underline{\sigma}_{>0}(B^T(x)L^{-1}(x))} \Big), \tag{4.9}$$

$$\Delta_{\dot{x}_r} = \Delta_f + \Delta_B(\|\mathbb{I}_m - C(s)\|_{\mathcal{L}_1}\Delta_h + \Delta_{u^*} + \rho\Delta_{\delta_u}), \tag{4.10}$$

$$\Delta_{\dot{x}} = \Delta_f + \Delta_B(2\Delta_h + \Delta_{u^*} + \rho\Delta_{\delta_u}), \tag{4.11}$$

$$\Delta_{\tilde{x}} = \sqrt{\frac{4\bar{\lambda}(P)\Delta_h(\Delta_{h_t} + \Delta_{h_x}\Delta_{\dot{x}})}{\underline{\lambda}(P)\underline{\lambda}(Q)} + \frac{4\Delta_h^2}{\underline{\lambda}(P)}}, \tag{4.12}$$

$$\Delta_{\tilde{\eta}} = (\Delta_{B_x^\dagger}\Delta_{\dot{x}} + (\|sC(s)\|_{\mathcal{L}_1} + \|A_m\|)\Delta_{B^\dagger})\Delta_{\tilde{x}}, \tag{4.13}$$

$$\Delta_\theta = \frac{\Delta_B \frac{c_3}{2}\Delta_{\tilde{\eta}}}{\lambda}, \tag{4.14}$$

$$\Delta_{\dot{\Psi}} = c_3\Delta_{B_x}\Delta_{\dot{x}}, \tag{4.15}$$

where $\mathcal{O}(\rho)$ is defined in Definition 4.1, Δ_{u^*} is defined in Assumption 3, $\Delta_f, \Delta_{f_x}, \Delta_B, \Delta_{B_x}$, are defined in Assumption 4; $\Delta_h, \Delta_{h_t}, \Delta_{h_x}$ are defined in Assumption 5; $\Delta_{B^\dagger}$ and $\Delta_{B_x^\dagger}$ are defined in Assumption 6; $\underline{\alpha}$ and $\bar{\alpha}$ are defined in Assumption 7; and $F(x)$ is defined as:

$$F(x) = -\frac{\partial W(x)}{\partial x}f(x) + \frac{\partial f(x)}{\partial x}W(x) + (\frac{\partial f(x)}{\partial x}W(x))^T + 2\lambda W(x),$$

where $W(x) = (\frac{\partial^2 V}{\partial x^2})^{-1}$ is referred to as the dual metric such that $L(x)^T L(x) = W(x)$ [17].

4.3.1 ILF Based Control: $u_c(t)$

Referring to [17], the following input feedback decomposition is considered:

$$u(t) = u_c(t) + u_a(t), \tag{4.16}$$

where $u_c(t)$ is the ILF based control designed to guarantee the incremental exponential stability of the nominal dynamics in Equation (4.3) and $u_a(t) \in \mathbb{R}^m$ is the $\mathscr{L}_1$ control signal, which will be discussed later.

Referring to [17], the following control law is proposed for $u_c(t)$:

$$u_c(t) = u^*(t) + k_c(x^*(t), x(t)), \tag{4.17}$$

where the law constructed in [17] is used for the feedback term, which is the solution to the following quadratic program:

$$k_c(x^*(t), x(t)) = \underset{k \in \mathbb{R}^m}{\arg\min} \|k\|^2, \tag{4.18}$$

$$s.t. \quad L_{\bar{F}(x,t)} V(x(t), x^*(t)) + L_{\bar{F}(x^*,t)} V(x(t), x^*(t)) \leq -2\lambda V(x(t), x^*(t)). \tag{4.19}$$

4.3.2 $\mathscr{L}_1$ Adaptive Control: $u_a(t)$

The calculation of the input signal $u_a(t)$ depends on three components, which are: the state-predictor, the adaptation law, and a low-pass filter. Similar to [17], the state-predictor is defined as:

$$d\hat{x} = [\bar{F}(x(t), u_c(t) + u_a(t) + \hat{\beta}_1(t)) + A_m \tilde{x}(t)]dt + [B(x(t)(\Sigma(x(t)) + \hat{\beta}_2(t)) + A_m \tilde{x}(t)]dw^*(t) \tag{4.20}$$

with $\hat{x}(0) = x_0$, and where $\hat{x}(t) \in \mathbb{R}^n$ is the state of the predictor, $\tilde{x}(t) = \hat{x}(t) - x(t)$ is the state prediction error, and $A_m \in \mathbb{R}^{n \times n}$ is an arbitrary Hurwitz matrix. Since now we have two uncertainties, $h(x,t)$ in the drift part and $\sigma(x,t)$ in the diffusion part, our state-predictor is written in the form of stochastic differential equation. The uncertainty estimates $\hat{\beta}_1(t), \hat{\beta}_2(t)$ in Equation (4.20) are governed by the following adaptation laws:

$$\dot{\hat{\beta}}_1(t) = \Gamma \underset{\mathcal{H}}{\text{Proj}}(\hat{\beta}_1(t), -B(x)^T P \tilde{x}(t)), \quad \beta_1(0) \in \mathcal{H}; \tag{4.21}$$

$$\dot{\hat{\beta}}_2(t) = \Gamma \underset{\mathcal{G}}{\text{Proj}}(\hat{\beta}_2(t), -B(x)^T P \tilde{x}(t)), \quad \beta_2(0) \in \mathcal{G}, \tag{4.22}$$

where $\Gamma > 0$ is the adaptation rate, $\mathcal{H} = \{y \in \mathbb{R}^m | \|y\| \leq \Delta_h\}$ is the set to which the uncertainty estimate $\hat{\beta}_1$ is restricted to remain in with Δ_h, defined in Assumption 5. Similarly, $\mathcal{G} = \{y \in \mathbb{R}^m | \|y\| \leq \Delta_{\Sigma_x}\}$ is the set to which the uncertainty estimate $\hat{\beta}_2$ is restricted to remain in with Δ_{Σ_x}, defined in Assumption 4. Furthermore, P is the positive definite matrix that solves the Lyapunov equation $A_m P + P A_m = -Q$ for some positive definite matrix Q. Moreover, $\text{Proj}_{\mathcal{H}}(\cdot,\cdot), \text{Proj}_{\mathcal{G}}(\cdot,\cdot)$ are the projection operator, standard in adaptive control literature [17].

Finally, the control law $u_a(t)$ is defined as the following Laplace transform:

$$u_a(s) = -C(s)(\hat{\beta}_1(s)), \tag{4.23}$$

where $C(s)$ is a low-pass filter with bandwidth ω and satisfies $C(0) = \mathbb{I}_m$.

4.3.3 Low-pass Filter Bandwidth and Adaptation Rate

The $\mathscr{L}_1$ adaptive controller uses a low-pass filter $C(s)$ to compensate the uncertainties in the stochastic system as in Equation (4.1) with $C(s) = \frac{\omega}{s+\omega}\mathbb{I}_m$. As will see in Chapter 5, the filter bandwidth and the adaptation rate become the factor in determining the bounds between trajectories and distributions behind

trajectories. However, the filter bandwidth ω and the adaptation rate Γ should satisfy a few conditions mentioned below. The reasons will be clear from the later derivations in Chapter 5.

Suppose Assumptions 1-8 hold; then for arbitrary positive numbers ρ_a, ϵ define:

$$\rho_r = \sqrt{\frac{\bar{\alpha}}{\underline{\alpha}}}\|x_0^* - x_0\| + \epsilon, \tag{4.24}$$

$$\rho = \rho_r + \rho_a, \tag{4.25}$$

$$\zeta_1(\omega) = \rho\Delta_B \frac{c_3}{\underline{\alpha}}\left(\frac{\Delta_h}{\mid 2\lambda - \omega \mid} + \frac{\Delta_{h_t} + \Delta_{h_x}\Delta_{\dot{x}_r}}{2\lambda\omega}\right), \tag{4.26}$$

$$\zeta_2(\omega) = c_3\left(\Delta_{B_x} + \Delta_B + \frac{\Delta_B}{\rho_1^2}\right)\left(\frac{\Delta_h}{\mid 2\lambda - \omega \mid} + \frac{\Delta_{h_t} + \Delta_{h_x}\Delta_{\dot{x}_r}}{2\lambda\omega}\right), \tag{4.27}$$

$$\zeta_3(\omega) = \Delta_{h_x}\frac{4\lambda\Delta_B c_3 + \Delta_{\dot{\Psi}} + (1 + \frac{1}{\rho_1^2})\Delta_B c_4 \Delta_{\dot{x}}}{\lambda\omega}, \tag{4.28}$$

where $\rho_1 = \rho + \rho_r$. $\Delta_{\dot{x}_r}, \Delta_{\dot{\Psi}}$ are defined in Equation (4.11), (4.15) respectively. Then, let the bandwidth ω of the low-pass filter $C(s)$ and the adaptation rate Γ satisfy the following conditions:

$$\rho_r^2 \geq \frac{V(x_0^*, x_0)}{\underline{\alpha}} + \zeta_1(\omega) + \frac{C_2 + C_3}{2\underline{\alpha}\lambda}, \tag{4.29}$$

$$\underline{\alpha} > (\zeta_2(\omega) + \zeta_3(\omega))\left(\frac{(\rho + \rho_r)^2}{\rho_a^2}\right), \tag{4.30}$$

$$\sqrt{\Gamma} > \frac{\Delta_\theta(\rho + \rho_a)}{\underline{\alpha}\rho_a^2 - (\zeta_2(\omega) + \zeta_3(\omega))(\rho + \rho_r)^2}, \tag{4.31}$$

where Δ_θ is another known positive number as in Equation (4.14).

Remark 4.2 *The functions $\zeta_1(\omega), \zeta_2(\omega)$, and $\zeta_3(\omega)$ are constructed such that all functions decrease with the increase in ω and converge to zero when ω goes to infinity. Thus, there always exists a way for Equations (4.29) − (4.31) to be satisfied by selecting a large enough bandwidth ω.*

Chapter 5

Analysis of the Controller Performance

In this chapter the performance of the uncertain stochastic system in Equation (4.1) with the $\mathscr{L}_1$ feedback control $u(t)$ defined in Equation (4.16) is analyzed. We will first define the reference system to derive the bounds between the ideal trajectory $x^*(t)$ and the state $x(t)$ of the uncertain stochastic system, and the bounds between ideal distributions ν_t^* and the distributions ν_t describing the state of the uncertain stochastic system. The analysis contains two steps: first the mean-square bounds between ideal trajectory $x^*(t)$ and the reference system $x_r(t)$, and the bounds for states between reference system $x_r(t)$ and actual system $x(t)$ are derived. Then we will derive the Wasserstein bounds between distributions for ideal trajectory ν_t^* and for reference system ν_{r_t}, and the Wasserstein bounds between distributions for states in reference system ν_{r_t} and for states in actual system ν_t. The proofs for every claim in this chapter are provided in Appendix B.

5.1 $\mathscr{L}_1$ Reference System

Assuming that all unknown uncertainties are known, we can analyze the stability and performance of the $\mathscr{L}_1$ adaptive controller when applied to the actual stochastic system in Equation (4.1) and provide bounds between the desired trajectory $x^*(t)$ and the state $x(t)$, similar to [17]. Based on this assumption, we first introduce the reference system:

$$dx_r(t) = \big[f(x_r(t)) + B(x_r(t))h(x_r(t),t)\big]dt + U_r(t) + B(x_r(t))(\Sigma(x_r(t)) + \sigma(x_r(t)))dw_r(t), \tag{5.1}$$

$$U_r(t) = [B(x_r(t))(u_r(t) - h_r(x_r(t)))]dt - B(x_r(t))\sigma_r(x_r(t))dw_r(t), \tag{5.2}$$

$$h_r(x_r(t),t) = C(s)h(x_r(t),t), \sigma_r(x_r(t)) = C(s)\sigma(x_r(t)), \tag{5.3}$$

$$x_r(0) = x_0^* \sim \nu_{r_0}, \tag{5.4}$$

where $U_r(t)$ is the reference input defined for the reference system and the $C(s) = \frac{\omega}{s+\omega}\mathbb{I}_m$. For the reference system, the Wiener process $w_r(t)$ is independent of the Wiener processes $w^*(t)$ and $w(t)$, and the same initial condition as in the ideal system is used. The reference system defines the best achievable performance through the cancellation of uncertainties within the bandwidth of the controller, defined through the low-pass filter.

5.2 Mean-square Stochastic Contraction Theorem

We start by obtaining the mean-square bound between ideal system trajectory $x^*(t)$ and the reference system state $x_r(t)$. Combining the reference system in Equation (5.1)-(5.4) and the ideal system in Equation (4.5), (4.6), we have:

$$\begin{cases} da(t) = \widehat{f}(a(t), \widehat{u}(t))dt + \widehat{\Sigma}(a(t))dW^2, \\ a(0) = (x^*(0), x_r(0))^T = (x_0^*, x_0^*)^T, \end{cases} \tag{5.5}$$

where

$$\widehat{f}(a(t), \widehat{u}(t)) = \begin{pmatrix} \bar{F}(x^*(t), u^*(t)) \\ F(x_r(t), -\eta_r(t)) \end{pmatrix},$$
$$\widehat{\Sigma}(a(t)) = \begin{pmatrix} B(x^*)\Sigma(x^*) & 0 \\ 0 & B(x_r)(\Sigma(x_r) - \sigma_r(x_r) + \sigma(x_r)) \end{pmatrix},$$
$$dW^2 = \begin{pmatrix} dw^*(t) \\ dw_r(t) \end{pmatrix}.$$

Define the shorthand notation: $v(x_r, t) = \Sigma(x_r) - \eta_r(t) + \sigma(x_r)$. With the IES Lyapunov function defined as $V(x^*(t), x_r(t))$, we have the following result:

Theorem 5.1 *Assume that the system* (5.5) *verifies Assumption 1-8. Let the initial condition x_0^* be independent of the noise and be given by a probability distribution $p(x_0^*)$. Then for any desired state trajectory $x^*(t)$ the state $x_r(t)$ of the reference system satisfies:*

$$\mathbb{E}(\|x^*(t) - x_r(t)\|^2) \leq \frac{1}{\underline{\alpha}}\left(\frac{C_2 + C_3}{2\lambda} + \underline{\alpha}\zeta_1(\omega)\right), \quad \forall t \geq 0. \tag{5.6}$$

Remark 5.1 *As one can see from the proof for Theorem 5.1 in Appendix B and from Definition 4.4, the following inequality holds:*

$$\underline{\alpha}\|x^*(t) - x_r(t)\|^2 \leq V(x^*, x_r) \leq \underline{\alpha}\zeta_1(\omega) + \frac{C_2 + C_3}{2\lambda}$$

under any initial condition pair $(x_0^*, x_0^*)^T$. Thus, we have:

$$\|x^*(t) - x_r(t)\|^2 \leq \zeta_1(\omega) + \frac{C_2 + C_3}{2\underline{\alpha}\lambda} \leq \rho_r^2$$

by the choice of ρ_r in Equation (4.29).

Next, we will compute the mean-square bound between the reference system in Equation (5.1)-(5.4) and the $\mathscr{L}_1$ closed-loop system in Equation (4.1) with (4.16). Combining the actual system and the reference system, we have:

$$\begin{cases} dy(t) = \bar{f}(y(t), \bar{u}(t))dt + \bar{\Sigma}(y(t))dW'^2 \\ y(0) = (x(0), x_r(0))^T = (x_0, x_0^*)^T, \end{cases} \tag{5.7}$$

where

$$\bar{f}(y(t)) = \begin{pmatrix} F(x(t), u(t)) \\ F(x_r(t), -\eta_r(t)) \end{pmatrix},$$
$$\bar{\Sigma}(y(t)) = \begin{pmatrix} B(x)(\Sigma(x) + \sigma(x)) & 0 \\ 0 & B(x_r)v(x_r, t)) \end{pmatrix},$$
$$dW'^2 = \begin{pmatrix} dw(t) \\ dw^*(t) \end{pmatrix},$$

$v(x_r, t)$ is the shorthand notation such that: $v(x_r, t) = \Sigma(x_r) - \eta_r(t) + \sigma(x_r)$. With the IES Lyapunov function defined as $V(x(t), x_r(t))$, we have the following result:

Theorem 5.2 *Assume that the system* (5.7) *verifies Assumption 1-8. Suppose that the stated assumptions and the conditions in Equations* (4.29) – (4.31) *hold. Then, let initial conditions x_0 and x_0^* be independent of the noise and given by a probability distribution $p(x_0, x_0^*)$. Additionally, under any given initial conditions pair $(x_0, x_r(0))^T$, assume that the trajectory of the $\mathscr{L}_1$ closed-loop system satisfies $x(t) \in \Omega(\rho, x^*(t))$, for all $t \in [0, \tau]$, for some $\tau > 0$, with $\Omega(\rho, x^*(t))$ and ρ defined in Definition 4.1 and Equation* (4.25)*, respectively. Then:*

$$\mathbb{E}(\|x(t) - x_r(t)\|^2) < \frac{1}{\underline{\alpha}}\big(e^{-2\lambda t}\mathbb{E}(V(x_0, x_0^*)) + \frac{C_1 + C_2}{2\lambda} + \underline{\alpha}\rho_a^2\big), \quad \forall t \in [0, \tau]. \tag{5.8}$$

Remark 5.2 *As one can see from proof of Theorem 5.2 in Appendix B, such $\underline{\alpha}$ is defined in Equation* (4.30) *to guarantee the following:*

$$\underline{\alpha}\rho_a^2 - (\zeta_2(\omega) + \zeta_3(\omega))(\rho + \rho_r)^2 > 0.$$

5.3 Stochastic Contraction in Wasserstein Sense

We now show the Wasserstein bounds between the distributions behind the trajectories in corresponding systems. The Wasserstein bounds are simple extensions from the mean-square bounds obtained in the previous part. The approach is borrowed and inspired by [12], with contraction results established in a mean-square sense for a class of stochastic systems. The formula of 2-Wasserstein distance is defined in Definition 3.3.

Another assumption required before proceeding is:

Assmpution 9. ([14], Condition (iii)) The diffusion function $B(x)(\Sigma(x,t) + \sigma(x,t))(\Sigma(x,t) + \sigma(x,t))^T B(x)^T$ satisfies:

$$y^T B(x)(\Sigma(x,t) + \sigma(x,t))(\Sigma(x,t) + \sigma(x,t))^T B(x)^T y \geq c y^T y, \qquad c > 0,$$

uniformly for all $y \in \mathbb{R}^n$, $t \in [0, T]$. Assumption 9 ensures the existence of a unique invariant distribution [14].

We now state the Wasserstein bound between the distributions behind trajectories in actual system and reference system first.

Theorem 5.3 *Given the actual system described in Equation* (4.1) *and the reference system described in* (5.1) – (5.4)*, satisfying Assumptions 1-9, the actual system and the reference system are stochastically contracting to each other in the sense that, for any pair of solutions $x(t)$, $x_r(t)$ with respective laws ν_t, ν_{r_t},*

we have:

$$\forall t \in [0,\tau], \qquad W_2(\nu_t, \nu_{r_t}) < (\frac{1}{\underline{\alpha}})^{\frac{1}{2}}\big(e^{-\lambda t}\sqrt{\bar{\alpha}}W_2(\nu_0, \nu_{r_0}) + \sqrt{\underline{\alpha}}\rho_a + \sqrt{\frac{C_1 + C_2}{2\lambda}}\big). \tag{5.9}$$

Now we will compute the Wasserstein bound between distributions behind trajectories in the ideal system and the reference system. Note that the ideal system and the reference system have the same initial condition distribution ($\nu_{r_0} = \nu_0^*$). But the distribution in ideal system ν_t^* and in the reference system ν_{r_t} will be different evolving with time. Then we have the following 2-Wasserstein bound between the ideal and the reference system:

$$W_2(\nu_t^*, \nu_{r_t}) \leq \sqrt{\frac{1}{\underline{\alpha}}}\Big(\sqrt{\frac{C_2 + C_3}{2\lambda}} + \sqrt{\underline{\alpha}\zeta_1(\omega)}\Big). \tag{5.10}$$

The proof is exactly identical to the proof for Theorem 5.3 and is thus omitted. Since the Wasserstein metric satisfies the triangle inequality [23], then the upper bound of the 2-Wasserstein distance between the distribution in the ideal and the actual systems is obtained:

$$\begin{aligned} W_2(\nu_t^*, \nu_t) &\leq W_2(\nu_t, \nu_{r_t}) + W_2(\nu_t^*, \nu_{r_t}) \\ &< \sqrt{\frac{1}{\underline{\alpha}}}\Big(\sqrt{\frac{C_1 + C_2}{2\lambda}} + \sqrt{\underline{\alpha}}\rho_a + \sqrt{\bar{\alpha}}W_2(\nu_0, \nu_{r_0})e^{-\lambda t}\Big) + \sqrt{\frac{1}{\underline{\alpha}}}\Big(\sqrt{\frac{C_2 + C_3}{2\lambda}} + \sqrt{\underline{\alpha}\zeta_1(\omega)}\Big) \\ &= \sqrt{\frac{1}{\underline{\alpha}}}\Big(\sqrt{\frac{C_1 + C_2}{2\lambda}} + \sqrt{\frac{C_2 + C_3}{2\lambda}} + \sqrt{\underline{\alpha}}(\sqrt{\zeta_1(\omega)} + \rho_a) + \sqrt{\bar{\alpha}}W_2(\nu_0, \nu_{r_0})e^{-\lambda t}\Big) \end{aligned} \tag{5.11}$$

Chapter 6

Numerical Simulations

Two illustrative examples are provided for the simulations. In the first example, we consider a feedback linearizable system adapted from [17] and design the controller for safe regulation around the equilibrium point. The uniform ultimate bounds are shown as discussed in Chapter 5, if the system were to start away from the equilibrium. In the second example, we refer to the same system and ensure a safe motion planning with desired trajectory tracking. Specifically, the effect of changing filter bandwidth and adaptation rate for determining the uniform bounds are shown.

6.1 Feedback Linearizable Systems

We consider the following stochastic system with the structure in Equation (1) which is adapted from Example 6.2 in [17] by with the addition of the noise:

$$f(x) = \begin{bmatrix} -x_1(t) + x_2(t) \\ -0.05x_2^3(t) - 3x_1(t) - x_2(t) \end{bmatrix}, B = \begin{bmatrix} 0.5 \\ -2 \end{bmatrix}, \tag{6.1}$$

where the state is defined as $x(t) = [x_1(t) \;\; x_2(t)]^T$. We choose the Riemannian Energy as the candidate for the Incremental Lyapunov Function V in the nominal system. As shown in [17], the Riemanninan Energy satisfies the properties of the Incremental Lyapunov Function. Thus, we use the dual metric and the associated convergence parameter similar to [17]:

$$W = \begin{bmatrix} 4.26 & -0.92 \\ -0.92 & 3.73 \end{bmatrix}, \qquad \lambda = 1.74 \tag{6.2}$$

Now suppose the system is under the sinusoidal disturbance, with the drift term $h(x,t) = 2\sin(2t)$ and the quadratic disturbance $\sigma(x(t)) = 0.01x_1^2(t)$ in the diffusion term, and the diffusion parameter is set to be $\Sigma(x(t)) = 0.01x_2^2(t)$. We choose the initial condition for the system sampled from the Multivariate Normal Distribution such that $\nu_0 = \mathcal{N}([1 \;\; 1]^T, \mathbb{I})$ with $x_0 = [0.354 \;\; -0.463]^T \sim \nu_0$. For the ideal system, we set the desired state as $x^* = [0 \;\; 0 \;\; 0]^T$. From [17], the desired state is also the equilibrium point of the system incidentally, which means that the desired control is $u^*(t) \equiv 0$. We implement the $\mathscr{L}_1$ adaptive controller in (5.1)-(5.4) and obtain simulation results for different values of the adaptation gain Γ and different values of

the filter bandwidth ω.

First we fix the adaptation rate to be $\Gamma = 4 \times 10^7$ and compare the Wasserstein bound between the distributions behind trajectories in the actual and ideal systems, as well as the Wasserstein bound between the distributions behind trajectories in the reference and ideal systems with different bandwidths ω. Figure

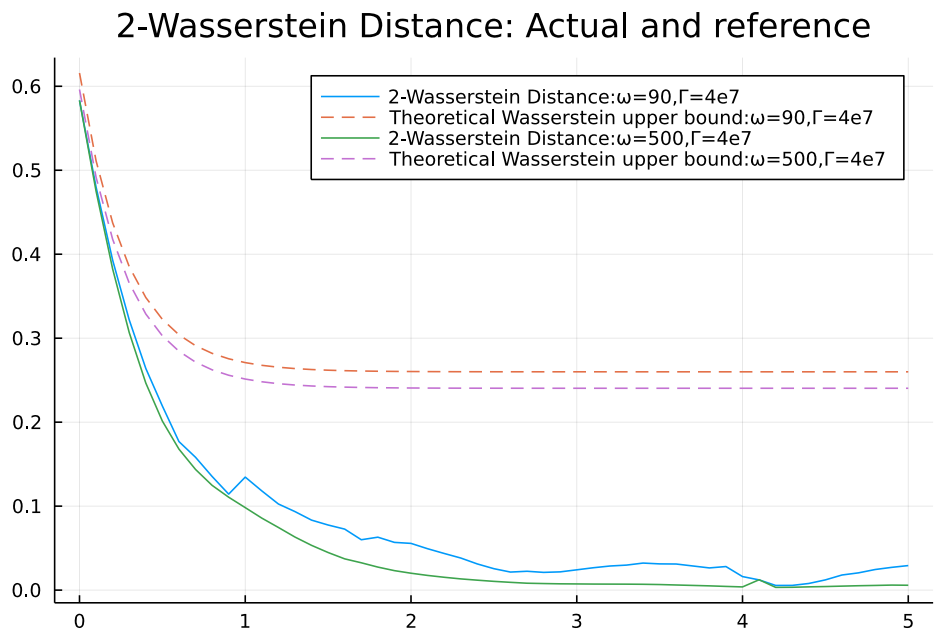

Figure 6.1: Comparison of 2-Wasserstein Distances between the actual system and the reference system using different bandwidths: $\omega = 90$ and $\omega = 500$. The dashed lines represent the 2-Wasserstein distance upper bounds as calculated from Equation (5.8) under different filter bandwidths ω.

6.1 shows the 2-Wasserstein distances and the 2-Wasserstein distance upper bounds between the actual and reference systems with the corresponding filter bandwidths ω. As seen from Figure 6.1, the injection of the Wiener process produces an oscillatory behavior between the states. Figure 6.2 shows the 2-Wasserstein distances and the 2-Wasserstein distance upper bounds between the reference and ideal systems under different values of ω respectively. From Figure 6.1 and Figure 6.2, we observe that the 2-Wasserstein distance falls under a smaller distance bound as the bandwidth ω increases. Figure 6.3 shows the 2-Wasserstein bound between the actual and ideal systems. It is observed that the 2-Wasserstein bound also decreases with the increase of the filter bandwidth ω. This satisfies our expectation because the increase in the bandwidth allows more cancellations for the uncertainties from both drift and diffusion parts in the system, and thus provides a better tracking performance in each of the systems.

Next we fix the filter bandwidth ω and compare the Wasserstein bound between the distributions behind the trajectories in the actual and the ideal systems, as well as the Wasserstein bound between the distributions behind the trajectories in the actual and reference systems with different adaptation rates Γ.

Figure 6.4 shows the 2-Wasserstein distances and the 2-Wasserstein upper bounds between the actual and reference systems, and Figure 6.5 shows the 2-Wasserstein distances and the 2-Wasserstein upper bounds between the actual and ideal systems under different values of Γ respectively. From Figure 6.4 and Figure 6.5, it is observed that the 2-Wasserstein distances fall under smaller 2-Wasserstein distance upper bounds, as the adaptation rate Γ increases. This satisfies our expectation because the adaptation rate plays the role in determining the bounds between the actual and reference systems as well as between the actual and ideal

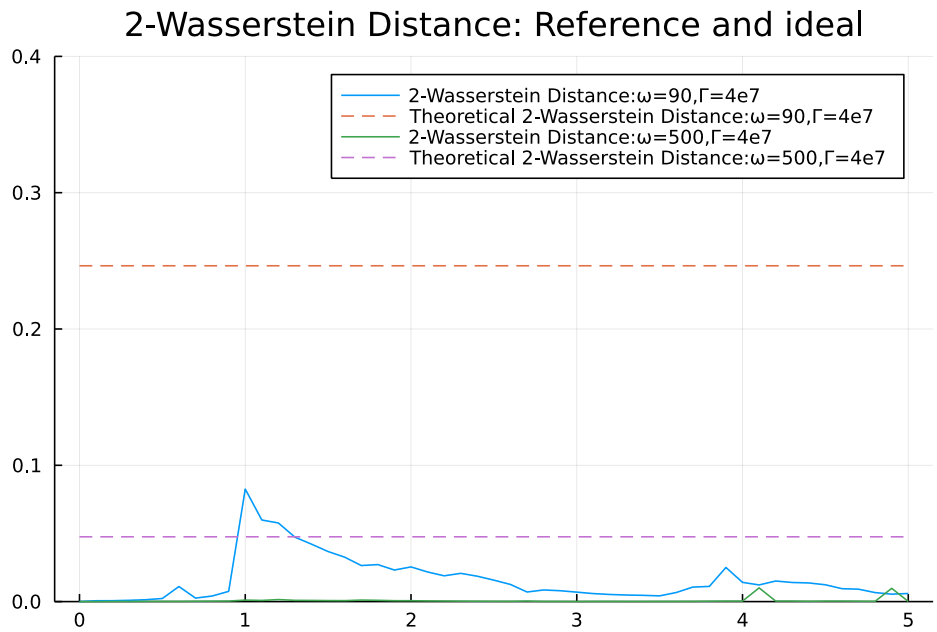

Figure 6.2: Comparison of 2-Wasserstein Distances between the reference system and the ideal system using different bandwidths: $\omega = 90$ and $\omega = 500$. The dashed lines represent the 2-Wasserstein distance upper bounds as calculated from Equation (5.6) under different filter bandwidths ω. It can be seen that the 2-Wasserstein distance for $\omega = 90$ exceeds the 2-Wasserstein distance upper bound for $\omega = 500$.

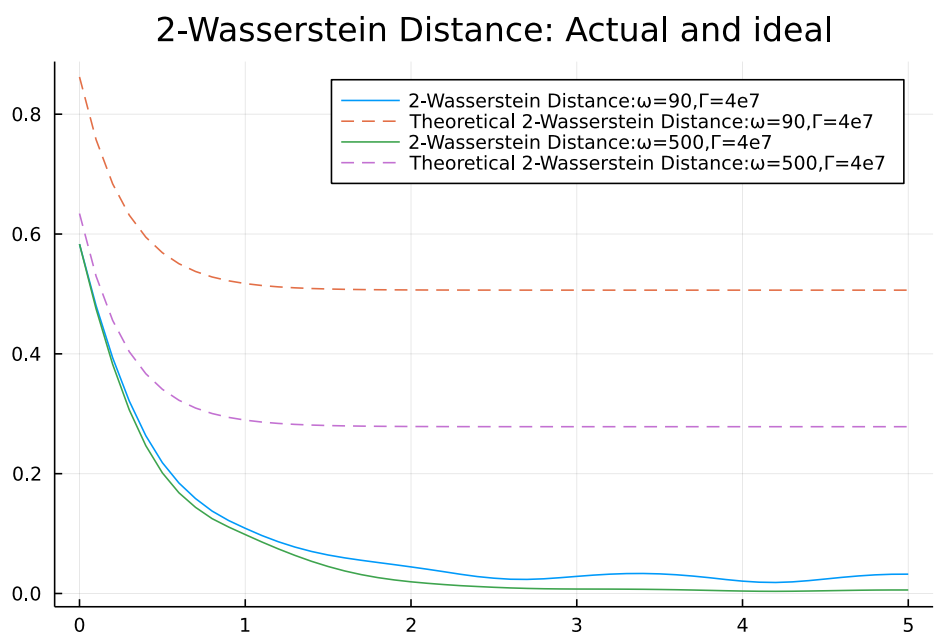

Figure 6.3: Comparison of 2-Wasserstein Distances between the actual system and the ideal system using different bandwidths: $\omega = 90$ and $\omega = 500$. The dashed lines represent the 2-Wasserstein distance upper bounds as calculated from Equation (5.11) under different filter bandwidths ω.

systems. From Equation (4.31), the adaptation rate is designed such that ρ_a is decreased with the increased Γ, and a reduced ρ_a should yield a smaller Wasserstein upper bound. Note that the increase in the adaptation

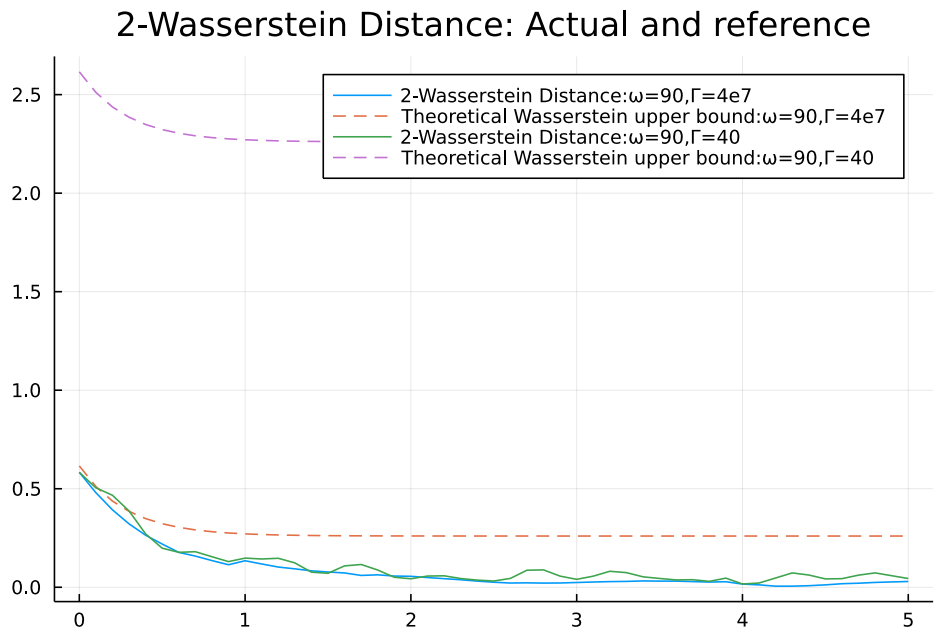

Figure 6.4: Comparison of 2-Wasserstein Distance between the actual system and the reference system using different adaptation rates: $\Gamma = 40$ and $\Gamma = 4 \times 10^7$. The dashed lines represent the 2-Wasserstein distance upper bounds as calculated from Equation (5.8) under different adaptation rates Γ. One can observe that the 2-Wasserstein distance for $\Gamma = 40$ exceeds the 2-Wasserstein distance upper bound for $\Gamma = 4 \times 10^7$.

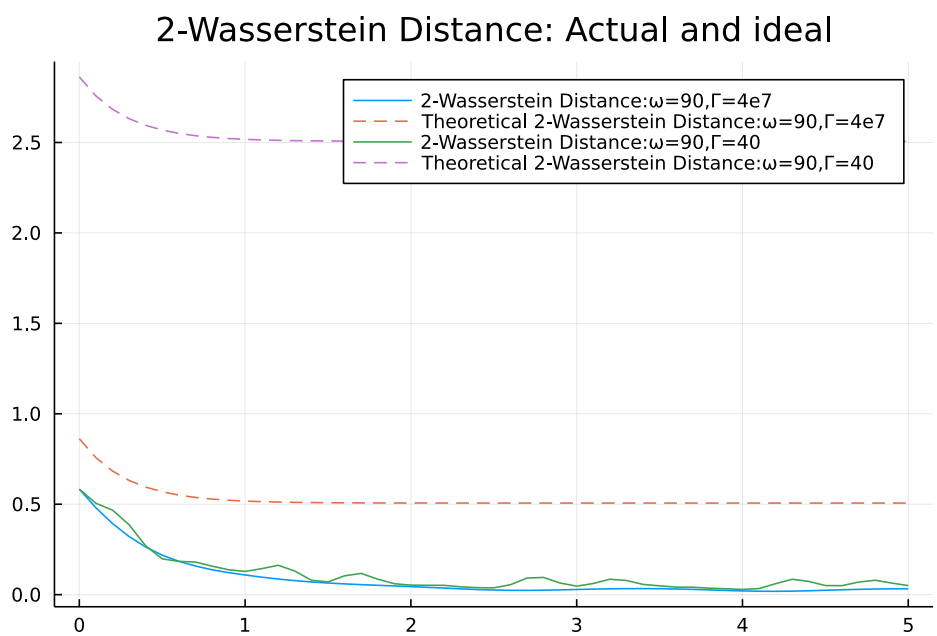

Figure 6.5: Comparison of 2-Wasserstein Distance between the actual system and the ideal system using different adaptation rates: $\Gamma = 40$ and $\Gamma = 4 \times 10^7$. The dashed lines represent the 2-Wasserstein distance upper bounds as calculated from Equation (5.11) under different adaptation rates Γ.

rate does not change the 2-Wasserstein bound between the reference and the ideal systems as one can see from Equation (5.6).

6.2 Safe Motion Planning: Collision Avoidance

We further observe the performance and robustness benefits using the IES Lyapunov Function with $\mathscr{L}_1$-adaptive control. Now the dynamic equation for our ideal system is:

$$dx^*(t) = (f(x^*(t)) + Bu^*(t))dt + \Sigma(x^*(t))dw^*(t)$$

with the dynamic equation for our actual system given as:

$$dx(t) = \big[f(x(t)) + B(u(t) + h(x,t))\big]dt + B(\Sigma(x(t)) + \sigma(x(t)))dw(t),$$

where f and B are the same as in Equation (6.1). The diffusion parameter Σ and the disturbance in the diffusion σ are the same as in Chapter 6.1. The disturbance in the drift term $h(x,t)$ remains the same with the convergence rate $\lambda = 1,74$. We first plan the ideal trajectory x^* and the control u^* using the iterative Linear Quadratic Gaussian (iLQG) algorithm while avoiding the designed obstacle [26], [27]. It is observed that using the control u^* from the planner, the trajectory from the actual system will be likely to hit the obstacle as shown in Figure 6.6. Due to the existence of noise in our systems, the trajectories shown here are sample trajectories.

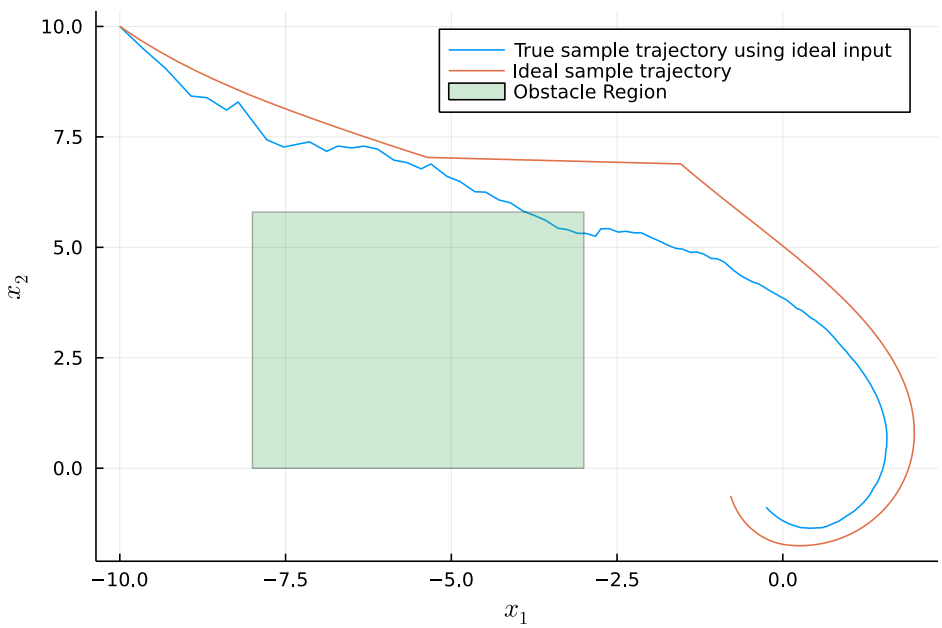

Figure 6.6: Comparison of sample trajectories between the ideal and the actual systems using the pure feedback from iterative Linear Quadratic Gaussian (iLQG). The planner is designed such that the ideal trajectories are at least 0.7 unit distance away from the obstacle with 99 percent confidence.

We then use the feedback from $\mathscr{L}_1$-adaptive control and obtain 1000 sample trajectories from the actual system with another 1000 sample trajectories from the ideal system. It is found that not only the system follows the ideal trajectory closely, but also is able to avoid the collision with obstacles through appropriate choice of the filter bandwidth ω and the adaptation rate Γ. The results are shown in Figure 6.7.

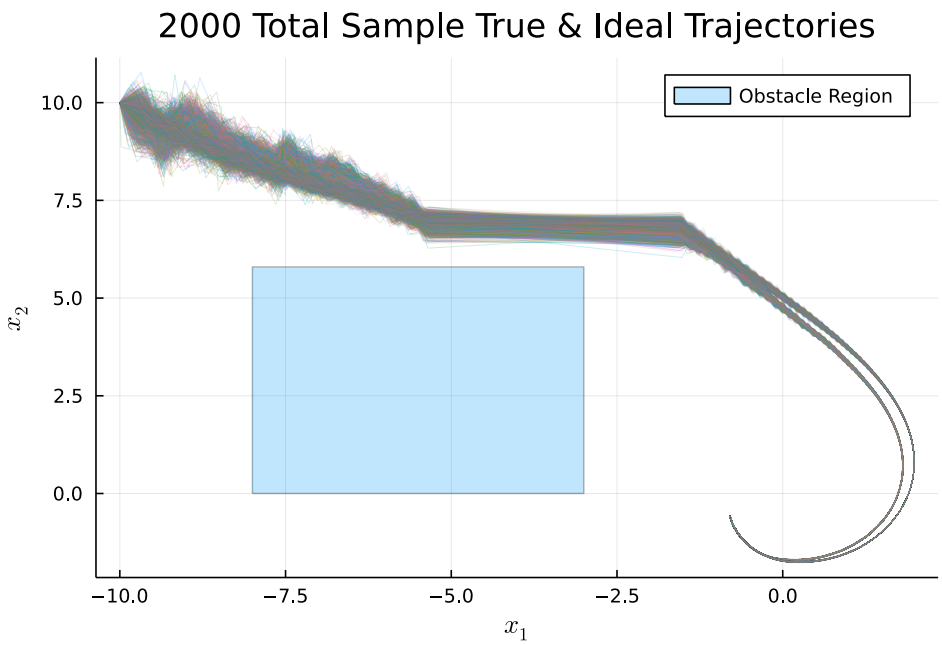

Figure 6.7: Comparison of sample trajectories between the ideal and the actual systems using $\mathscr{L}_1$-adaptive control. The grey region represents the overlapping between the trajectories with highest frequency.

Chapter 7

Conclusion

7.1 Summary

This thesis considers nonlinear stochastic systems with unknown uncertainties and parameters driven by Wiener processes and uses $\mathscr{L}_1$ adaptive control theory to ensure the robustness of the system. The presented work uses an incremental Lyapunov function with $\mathscr{L}_1$ adaptive control. The performance of the closed-loop system using the proposed $\mathscr{L}_1$ adaptive controller is evaluated. The system is shown to provide uniform bounds for trajectories and corresponding distributions behind trajectories, which act as safety-certificates. Also, it is shown that the uniform bounds can be adjusted by changing the adaptation rate and the filter bandwidth. Simulations are provided to illustrate the theoretical results.

7.2 Future Work

It is found that the choice of the diffusion part affects the performance for our trajectory solvers. For example, if the diffusion part of our stochastic system is too large, then the solution will be numerically unstable by observation. Future work will include the incorporation of stochastic stability for the nonlinear stochastic systems to guarantee the performance for the solvers.

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Appendix A

Technical Results

Lemma A.1 *Let the state $x_r(t)$ of the reference system in Equations (5.1) − (5.4) and the state $x(t)$ of the real system in Equation (4.1) with control input $u(t)$ in Equation (4.14) satisfy $x_r(t), x(t) \in \Omega(\rho, x^*(t))$ for all $t \in [0, \tau]$, for some $\tau > 0$. Additionally, let Assumptions 3-5 hold. Then the following inequality is satisfied:*

$$\|\frac{d}{dt}(B(x)^T \frac{\partial V}{\partial x}^T)\| \leq \Delta_{\dot{\Psi}} \|x_r(t) - x(t)\| + 2\Delta_B c_4 \Delta_{\dot{x}}.$$

Proof. We apply chain rule and triangle inequality to obtain

$$\begin{aligned}
\|\frac{d}{dt}(B(x)^T \frac{\partial V}{\partial x}^T)\| &\leq \|\frac{\partial B}{\partial x}^T \dot{x} \frac{\partial V}{\partial x}^T\| + \|B(x)^T \frac{d}{dt}(\frac{\partial V}{\partial x}^T)\| \\
&= \|\frac{\partial B}{\partial x}^T \dot{x} \frac{\partial V}{\partial x}^T\| + \|B(x)(\frac{\partial}{\partial x}\frac{\partial V}{\partial x})\dot{x}\| \\
&= \|\frac{\partial B}{\partial x}^T \dot{x} \frac{\partial V}{\partial x}^T\| + \|B(x)\frac{\partial^2 V}{\partial x^2}\dot{x}\|.
\end{aligned}$$

From Assumptions 3-5 and Lemma A.7 in [17], we have the following result:

$$\|\frac{\partial B}{\partial x}^T \dot{x} \frac{\partial V}{\partial x}^T\| \leq \Delta_{B_x} \Delta_{\dot{x}} c_3 \|x_r(t) - x(t)\|, \quad \|B(x)\frac{\partial^2 V}{\partial x^2}\dot{x}\| \leq 2\Delta_B c_4 \Delta_{\dot{x}}.$$

Thus, we have:

$$\begin{aligned}
\|\frac{d}{dt}(B(x)^T \frac{\partial V}{\partial x}^T)\| &\leq c_3 \Delta_{B_x} \Delta_{\dot{x}} \|x_r(t) - x(t)\| + 2\Delta_B c_4 \Delta_{\dot{x}}. \\
&= \Delta_{\dot{\Psi}} \|x_r(t) - x(t)\| + 2\Delta_B c_4 \Delta_{\dot{x}}.
\end{aligned}$$

□

Appendix B

Main Results

The proof for Theorem 5.1 is given below:

Proof. Let $a(t) = (x^*(t), x_r(t))^T$. Using the infinitesimal operator $\mathcal{L}$ [22] for the combined system (5.5), we have:

$$\begin{aligned}\mathcal{L}V(a) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial a}\widehat{f}(a,\widehat{u}) + \frac{1}{2}tr\big(\widehat{\Sigma}(a(t))^T\frac{\partial^2 V}{\partial a^2}\widehat{\Sigma}(a(t))\big) \\ &\leq \dot{V}(x^*(t), x_r(t)) + C_2 + C_3,\end{aligned} \tag{B.1}$$

where

$$\begin{aligned}&\frac{1}{2}tr\big(\widehat{\Sigma}(a(t))^T\frac{\partial^2 V}{\partial a^2}\widehat{\Sigma}(a(t))\big) \\ &= \frac{1}{2}tr\big(\Sigma(x^*)^T B(x^*)^T\frac{\partial^2 V}{\partial x^{*2}}B(x^*)\Sigma(x^*)\big) + \frac{1}{2}tr\big(v(x_r,t)^T B(x_r)^T\frac{\partial^2 V}{\partial x_r^2}B(x_r)v(x_r,t)\big) \\ &\leq C_2 + C_3,\end{aligned}$$

given by Assumption 8. Consider the time derivative of $V(x^*(t), x_r(t))$:

$$\dot{V}(x^*(t), x_r(t)) = \frac{\partial V}{\partial x_r}\big[f(x_r) + B(x_r)(u_{c,r}(t) - \eta_r(t) + h(x_r,t))\big] + \frac{\partial V}{\partial x^*}\big(f(x^*) + B(x^*)u^*(t)\big). \tag{B.2}$$

Since the ideal nominal system admits an IES Lyapunov function, then by Definition 4.4, the following holds:

$$\begin{aligned}L_{\bar{F}(x_r,u_{c,r})}V(x^*,x_r) + L_{\bar{F}(x^*,u^*)}V(x^*,x_r) &= \frac{\partial V}{\partial x_r}\big(f(x_r) + B(x_r)u_{c,r}\big) + \frac{\partial V}{\partial x^*}\big(f(x^*) + B(x^*)u^*\big) \\ &\leq -2\lambda V(x^*,x_r).\end{aligned}$$

Since C_2 and C_3 are both nonnegative, Equation (B.2) can be rewritten as:

$$\begin{aligned}\dot{V}(x^*,x_r) &\leq \dot{V}(x^*,x_r) + C_2 + C_3 \\ &\leq -2\lambda V(x^*,x_r) + \frac{\partial V}{\partial x_r}B(x_r)(h(x_r,t) - \eta_r(t)) + C_2 + C_3.\end{aligned}$$

Integrating both sides above, we have:

$$V(x^*, x_r) \leq e^{-2\lambda t} V(x_0^*, x_0^*) + (C_2 + C_3) \int_0^t e^{2\lambda(\nu - t)} d\nu + \int_0^t e^{-2\lambda(t-\nu)} \frac{\partial V}{\partial x_r} B(x_r(\nu))(h(x_r(\nu), \nu) - \eta(\nu)) d\nu$$
$$\leq \int_0^t e^{-2\lambda(t-\nu)} \frac{\partial V}{\partial x_r} B(x_r(\nu))(h(x_r(\nu), \nu) - \eta_r(\nu)) d\nu + \frac{C_2 + C_3}{2\lambda} \tag{B.3}$$

with $V(x_0^*, x_0^*) = 0$. The right-hand side of the integral term can be expressed as the solution to the following virtual scalar system [17]:

$$\dot{w}(t) = -2\lambda w(t) + \frac{\partial V}{\partial x_r} B(x_r(t)) \xi(t), \quad w(0) = 0, \tag{B.4}$$

$$\xi(s) = (1 - C(s)) \mathscr{L}[h(x_r, t)]. \tag{B.5}$$

With Assumptions 2, 4, 5, 8 and Lemma A.6 [17], we have the following:

$$\|\frac{\partial V}{\partial x_r} B(x_r)\| \leq \rho c_3 \Delta_B, \qquad \|h(x_r, t)\| \leq \Delta_h,$$
$$\|\dot{h}(x_r, t)\| = \|\frac{\partial h(x_r, t)}{\partial t} + \frac{\partial h(x_r, t)}{\partial x_r} \dot{x}_r\|$$
$$\leq \Delta_{h_t} + \Delta_{h_x} \Delta_{\dot{x}_r}.$$

Then the solution of a linear system of the form in Equations (B.4) and (B.5) satisfies the following norm bound from Lemma A.1 [17]:

$$\|w(t)\| \leq \rho c_3 \Delta_B (\frac{\Delta_h}{\mid 2\lambda - \omega \mid} + \frac{\Delta_{h_t} + \Delta_{h_x} \Delta_{\dot{x}_r}}{2\lambda\omega}) = \underline{\alpha} \zeta_1(\omega),$$

where $\zeta_1(\omega)$ is defined in Equation (4.26). Thus, Equation (B.3) can be written as:

$$V(x^*, x_r) \leq \underline{\alpha} \zeta_1(\omega) + \frac{C_2 + C_3}{2\lambda}. \tag{B.6}$$

Taking the expectation under the initial condition $a_0 = (x_0^*, x_0^*)^T$, we have:

$$\mathbb{E}_{a_0} V(x^*, x_r) \leq \mathbb{E}_{a_0} [\underline{\alpha} \zeta_1(\omega) + \frac{C_2 + C_3}{2\lambda}]$$
$$= \underline{\alpha} \zeta_1(\omega) + \frac{C_2 + C_3}{2\lambda}.$$

Integrate with respect to $p(x_0^*)$:

$$\mathbb{E}\big(V(x^*(t), x_r(t))\big) \leq \underline{\alpha} \zeta_1(\omega) + \frac{C_2 + C_3}{2\lambda}.$$

By Definition 4.4, we have:

$$\underline{\alpha} \|x^*(t) - x_r(t)\|^2 \leq V(x^*(t), x_r(t)).$$

Thus, we finally have:

$$\begin{aligned}\mathbb{E}(\|x^*(t)-x_r(t)\|^2) &\le \frac{1}{\underline{\alpha}}\mathbb{E}\big(V(x^*(t),x_r(t))\big)\\ &\le \frac{1}{\underline{\alpha}}\Big(\frac{C_2+C_3}{2\lambda}+\underline{\alpha}\zeta_1(\omega)\Big).\end{aligned}$$

□

We now state the proof for Theorem 5.2.

Proof. Let $y(t)=(x(t),x_r(t))^T$. Now fix the initial condition $y(0)$. By all assumptions and results from Theorem 5.1 with Remark 5.1, under the initial condition pair $(x_0,x_0^*)^T$, we have the following:

$$\begin{aligned}\|x(t)-x_r(t)\| &= \|x(t)-x^*(t)+x^*(t)-x_r(t)\|\\ &\le \|x(t)-x^*(t)\|+\|x^*(t)-x_r(t)\|\\ &= \rho+\rho_r,\end{aligned}$$

which means:

$$\|x(t)-x_r(t)\|^2 \le (\rho+\rho_r)^2=\rho_1^2.$$

Using the infinitesimal operator $\mathcal{L}$ for the combined system, we have:

$$\begin{aligned}\mathcal{L}V(y) &= \frac{\partial V}{\partial t}+\frac{\partial V}{\partial y}\bar{f}(y,\bar{u})+\frac{1}{2}tr\big(\bar{\Sigma}(y(t))^T\frac{\partial^2 V}{\partial y^2}\bar{\Sigma}(y(t))\big)\\ &\le \dot{V}(x(t),x_r(t))+C_1+C_2,\end{aligned} \tag{B.7}$$

where

$$\begin{aligned}&\frac{1}{2}tr\big(\bar{\Sigma}(y(t))^T\frac{\partial^2 V}{\partial y^2}\bar{\Sigma}(y(t))\big)\\ &=\frac{1}{2}tr\big((\Sigma(x)+\sigma(x))^TB(x)^T\frac{\partial^2 V}{\partial x^2}B(x)(\Sigma(x)+\sigma(x))\big)+\frac{1}{2}tr\big(v(x_r,t)^TB(x_r)^T\frac{\partial^2 V}{\partial x_r^2}B(x_r)v(x_r,t)\big)\\ &\le C_1+C_2.\end{aligned}$$

Consider the time derivative of $V(x(t),x_r(t))$:

$$\dot{V}(x(t),x_r(t))=\frac{\partial V}{\partial x_r}\big[f(x_r)+B(x_r)(u_{c,r}(t)-\eta_r(t)+h(x_r,t))\big]+\frac{\partial V}{\partial x}\big(f(x)+B(x)(u_c(t)+h(x,t)-\hat{\eta}(t)\big). \tag{B.8}$$

By Definition 4.4 of IES Lyapunov Function, we have:

$$\begin{aligned}L_{\bar{F}(x_r,u_{c,r})}V(x,x_r)+L_{\bar{F}(x,u)}V(x,x_r) &= \frac{\partial V}{\partial x_r}\big(f(x_r)+B(x_r)u_{c,r}(t)\big)+\frac{\partial V}{\partial x}\big(f(x)+B(x)u(t)\big)\\ &\le -2\lambda V(x,x_r).\end{aligned}$$

Since C_1 and C_2 are both nonnegative, Equation (B.8) can be rewritten as:

$$\begin{aligned}\dot{V}(x(t), x_r(t)) &\leq \dot{V}(x(t), x_r(t)) + C_1 + C_2 \\ &\leq -2\lambda V(x(t), x_r(t)) + \frac{\partial V}{\partial x_r} B(x_r)(h(x_r, t) - \eta_r(t)) + \frac{\partial V}{\partial x} B(x)(h(x, t) - \hat{\eta}(t)) + C_1 + C_2 \\ &= -2\lambda V(x, x_r) + \Psi(x_r)^T (h(x_r, t) - \eta_r(t)) - \Psi(x)^T (h(x, t) - \hat{\eta}(t)) + C_1 + C_2,\end{aligned}$$

where $\Psi(x_r) := B(x_r)^T [\frac{\partial V}{\partial x_r}]^T$ and $\Psi(x) := -B(x)^T [\frac{\partial V}{\partial x}]^T$ are introduced for clarity. Define $\eta(s) = C(s)\mathscr{L}[h(x, t)]$. By adding and subtracting $\Psi(x)^T (h(x_r, t) - \eta_r(t) + \eta(t))$ on the right-hand side, we obtain

$$\begin{aligned}\dot{V}(x, x_r) \leq &-2\lambda V(x, x_r) + (\Psi(x_r) - \Psi(x))^T (h(x_r, t) - \eta_r(t)) \\ &+ \Psi(x)^T (h(x_r, t) - \eta_r(t) - h(x, t) + \eta(t)) \\ &+ \Psi(x)^T (\hat{\eta}(t) - \eta(t)) + C_1 + C_2.\end{aligned}$$

Since $h(x_r, t) - \eta_r(t) = \mathscr{L}^{-1}[(1 - C(s))\mathscr{L}[h(x_r, t)]]$, $h(x, t) - \eta(t) = \mathscr{L}^{-1}[(1 - C(s))\mathscr{L}[h(t, x)]]$, with $\tilde{\eta}(t) = \hat{\eta}(t) - \eta(t)$, we rewrite the equation above as

$$\dot{V}(x, x_r) \leq -2\lambda V(x, x_r) + \Phi_1(x_r, x) + \Phi_2(x_r, x) + \Phi_3(x_r, x) + C_1 + C_2, \tag{B.9}$$

where

$$\begin{aligned}\Phi_1(x_r, x) &:= (\Psi(x_r) - \Psi(x))^T \mathscr{L}^{-1}[(1 - C(s))\mathscr{L}[h(t, x_r)]], \\ \Phi_2(x_r, x) &:= \Psi(x)^T \mathscr{L}^{-1}[(1 - C(s))\mathscr{L}[h(t, x_r) - h(t, x)]], \\ \Phi_3(x_r, x) &:= \Psi(x)^T \tilde{\eta}(t).\end{aligned}$$

Solving the differential equation in Equation (B.9), we obtain:

$$\begin{aligned}V(x, x_r) &\leq e^{-2\lambda t} V(x_0, x_0^*) + (C_1 + C_2) \int_0^t e^{-2\lambda(t-\nu)} d\nu + \int_0^t e^{-2\lambda(t-\nu)} (\Phi_1(x_r, x) + \Phi_2(x_r, x) + \Phi_3(x_r, x)) d\nu \\ &\leq e^{-2\lambda t} V(x_0, x_0^*) + \frac{C_1 + C_2}{2\lambda} + \int_0^t e^{-2\lambda(t-\nu)} (\Phi_1(x_r, x) + \Phi_2(x_r, x) + \Phi_3(x_r, x)) d\nu.\end{aligned} \tag{B.10}$$

Notice that $\|\Psi(x_r) - \Psi(x)\|$ satisfies the following bound:

$$\|\Psi(x_r) - \Psi(x)\| \leq \|B(x_r)^T \frac{\partial V}{\partial x_r}^T + B(x)^T \frac{\partial V}{\partial x}^T\|.$$

Adding and subtracting $B(x)^T \frac{\partial V}{\partial x_r}^T$ from the right hand side of the equation above, we obtain

$$\|\Psi(x_r) - \Psi(x)\| \leq \|(B(x_r) - B(x))^T \frac{\partial V}{\partial x_r}^T + B(x)^T (\frac{\partial V}{\partial x_r}^T + \frac{\partial V}{\partial x}^T)\|. \tag{B.11}$$

Since $x_r(t) \in \Omega(\rho_r, x^*(t))$ from Remark 5.1 and $x(t) \in \Omega(\rho, x^*(t))$ for all $t \in [0, \tau^*]$ by assumptions in Theorem 5.2, the following bounds hold for $t \in [0, \tau^*]$ as a result of Assumption 4:

$$\|B(x_r) - B(x)\| \leq \Delta_{B_x} \|x_r - x\|, \qquad \|B(x)\| \leq \Delta_B.$$

By Assumption 8, we have:

$$\|\frac{\partial V}{\partial x_r}\| \leq c_3\|x_r(t) - x(t)\|, \qquad \|\frac{\partial V}{\partial x_r} + \frac{\partial V}{\partial x}\| \leq 2c_3\|x_r(t) - x(t)\| \leq c_3\|x_r(t) - x(t)\|^2 + c_3.$$

Substituting these bounds in Equation (B.10) produces

$$\|\Psi(x_r) - \Psi(x)\| \leq (c_3\Delta_{B_x} + c_3\Delta_B)\|x_r(t) - x(t)\|^2 + c_3\Delta_B, \tag{B.12}$$

which holds for all $t \in [0, \tau^*]$. Additionally, from Assumption 5 the following inequalities hold:

$$\|h(x_r, t)\| \leq \Delta_h, \|h(x_r, t) - h(x, t)\| \leq \Delta_{h_x}\|x_r(t) - x(t)\|. \tag{B.13}$$

Since $\|\frac{\partial h}{\partial t}(t, x_r)\| \leq \Delta_{h_t}$ and $\|\frac{\partial h}{\partial x}(t, x_r)\| \leq \Delta_{h_x}$ from Assumption 5, and $\|\dot{x}_r\|_{L_\infty} \leq \Delta_{\dot{x}_r}$ from Lemma A.6 [17], the following inequality is satisfied

$$\|\dot{h}(t, x_r)\| = \|\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x_r}\dot{x}_r\| \leq \Delta_{h_t} + \Delta_{h_x}\Delta_{\dot{x}_r}. \tag{B.14}$$

Since $\|\frac{\partial V}{\partial x}\| \leq c_3\|x_r(t) - x(t)\|$ for all $t \in [0, \tau^*]$, the following holds

$$\|\Psi(x)\| = \|B(x)^T\frac{\partial V}{\partial x}^T\| \leq \Delta_B c_3\|x_r(t) - x(t)\| \tag{B.15}$$

for all $t \in [0, \tau^*]$. Using Lemma A.1 from Appendix A, the following result holds for all $t \in [0, \tau^*]$

$$\|\dot{\Psi}(x)\| \leq \Delta_{\dot{\Psi}}\|x_r(t) - x(t)\| + 2\Delta_B c_4\Delta_{\dot{x}}. \tag{B.16}$$

In order to derive bounds on Equation (B.10), define the following scalar trajectories

$$z_1(t) = \int_0^t e^{-2\lambda(t-\nu)}\Phi_1(x, x_r)d\nu, \qquad z_2(t) = \int_0^t e^{-2\lambda(t-\nu)}\Phi_2(x, x_r)d\nu.$$

Then, the functions $z_i, i \in \{1, 2\}$, are the states of the following system

$$\dot{z}_i(t) = -2\lambda z_i(t) + b_i(t)\xi_i(t), \qquad z_i(0) = z_0. \tag{B.17}$$

$$\xi_i(s) = (1 - C(s))\sigma_i(s), \tag{B.18}$$

where

$$b_1(t) = \Psi(x_r) - \Psi(x), \qquad \sigma_1(t) = h(t, x_r),$$
$$b_2(t) = \Psi(x), \qquad \sigma_2(t) = h(t, x_r) - h(t, x).$$

In previous discussion, we assumed that:

$$\|x(t) - x_r(t)\|^2 \leq \rho_1^2.$$

Using Lemma A.1 [17] for the $z_1(t)$ system, Lemma A.2 [17] for the $z_2(t)$ system, and the bounds in Equation

(B.11) to (B.15), we have the following inequalities

$$
\begin{aligned}
\|z_1(t)\| &\leq \|\Psi(x_r) - \Psi(x)\|(\frac{\Delta_h}{\mid 2\lambda - \omega \mid} + \frac{\Delta_{h_t} + \Delta_{h_x}\Delta_{\dot{x}_r}}{2\lambda\omega}) \\
&\leq \big((c_3\Delta_{B_x} + c_3\Delta_B)\|x_r - x\|^2 + c_3\Delta_B\big)(\frac{\Delta_h}{\mid 2\lambda - \omega \mid} + \frac{\Delta_{h_t} + \Delta_{h_x}\Delta_{\dot{x}_r}}{2\lambda\omega}) \\
&\leq \big((c_3\Delta_{B_x} + c_3\Delta_B)\rho_1^2 + c_3\Delta_B\big)(\frac{\Delta_h}{\mid 2\lambda - \omega \mid} + \frac{\Delta_{h_t} + \Delta_{h_x}\Delta_{\dot{x}_r}}{2\lambda\omega}) \\
&= \rho_1^2(c_3)(\Delta_{B_x} + \Delta_B + \frac{\Delta_B}{\rho_1^2})(\frac{\Delta_h}{\mid 2\lambda - \omega \mid} + \frac{\Delta_{h_t} + \Delta_{h_x}\Delta_{\dot{x}_r}}{2\lambda\omega}) \\
&= \rho_1^2\zeta_2(\omega),
\end{aligned} \tag{B.19}
$$

and

$$
\begin{aligned}
\|z_2(t)\| &\leq \|h(x_r,t) - h(x,t)\|\frac{4\lambda\|\Psi(x)\| + \|\dot{\Psi}(x)\|}{\lambda\omega} \\
&\leq \Delta_{h_x}\|x_r(t) - x(t)\|\frac{4\lambda\Delta_B c_3\|x_r(t) - x(t)\| + \|\dot{\Psi}(x)\|}{\lambda\omega} \\
&\leq \Delta_{h_x}\rho_1\frac{4\lambda\Delta_B c_3\rho_1 + \Delta_{\dot{\Psi}}\rho_1 + 2\Delta_B c_4\Delta\dot{x}}{\lambda\omega} \\
&= \Delta_{h_x}\frac{4\lambda\Delta_B c_3 + \Delta_{\dot{\Psi}}}{\lambda\omega}\rho_1^2 + \Delta_{h_x}\frac{2\Delta_B c_4\Delta_{\dot{x}}}{\lambda\omega}\rho_1 \\
&\leq \Delta_{h_x}\frac{4\lambda\Delta_B c_3 + \Delta_{\dot{\Psi}}}{\lambda\omega}\rho_1^2 + \Delta_{h_x}\frac{\Delta_B c_4\Delta_{\dot{x}}}{\lambda\omega}(\rho_1^2 + 1) \\
&= \Delta_{h_x}\frac{4\lambda\Delta_B c_3 + \Delta_{\dot{\Psi}} + (1 + \frac{1}{\rho_1^2})\Delta_B c_4\Delta_{\dot{x}}}{\lambda\omega}\rho_1^2 \\
&= \rho_1^2\zeta_3(\omega)
\end{aligned} \tag{B.20}
$$

for all $t \in [0, \tau^*]$, and where ζ_2 and ζ_3 are defined in Equations (4.27) and (4.28) respectively. Moreover, it is easy to show from Equation (A.21) [17] that

$$
\|\int_0^t e^{-2\lambda(t-\nu)}\Phi_3(x_r, x)d\nu\| \leq \frac{\Delta_\theta\rho_1}{\sqrt{\Gamma}} \tag{B.21}
$$

for all $t \in [0, \tau^*]$, where Δ_θ is defined in Equation (15). Substituting Equation (B.19), (B.20) and (B.21) into Equation (B.10) we obtain the following bound:

$$
V(x, x_r) \leq e^{-2\lambda t}V(x_0, x_0^*) + \rho_1^2\zeta_2(\omega) + \rho_1^2\zeta_3(\omega) + \frac{\Delta_\theta\rho_1}{\sqrt{\Gamma}} + \frac{C_1 + C_2}{2\lambda}
$$

for all $t \in [0, \tau^*]$. Note that from Equation (4.31) the adaptation rate Γ is chosen such that

$$
\begin{aligned}
\sqrt{\Gamma} &> \frac{\Delta_\theta(\rho + \rho_r)}{\underline{\alpha}\rho_a^2 - (\zeta_2(\omega) + \zeta_3(\omega))(\rho + \rho_r)^2} \\
&= \frac{\Delta_\theta\rho_1}{\underline{\alpha}\rho_a^2 - (\zeta_2(\omega) + \zeta_3(\omega))\rho_1^2}.
\end{aligned}
$$

Plugging into the above equation, we have:

$$V(x, x_r) < e^{-2\lambda t} V(x_0, x_0^*) + \underline{\alpha}\rho_a^2 + \frac{C_1 + C_2}{2\lambda}.$$

Take the expectation operator under the initial condition $y(0)$ and then integrate with respect to $p(x_0, x_0^*)$; then we have:

$$\forall t \in [0, \tau] \quad \mathbb{E}(V(x, x_r)) < \frac{1}{\underline{\alpha}}\Big(e^{-2\lambda t}\mathbb{E}\big(V(x_0, x_0^*)\big) + \frac{C_1 + C_2}{2\lambda} + \underline{\alpha}\rho_a^2\Big).$$

By Definition 4.4, we have:

$$\underline{\alpha}\|x(t) - x_r(t)\|^2 \leq V(x(t), x_r(t)).$$

Thus, we finally have:

$$\forall t \in [0, \tau] \quad \mathbb{E}(\|x(t) - x_r(t)\|^2) < \frac{1}{\underline{\alpha}}\Big(e^{-2\lambda t}\mathbb{E}\big(V(x_0, x_0^*)\big) + \frac{C_1 + C_2}{2\lambda} + \underline{\alpha}\rho_a^2\Big).$$

□

We now show the proof for Theorem 5.3.

Proof. From Theorem 5.2, Equation (5.8) gives:

$$\begin{aligned}\mathbb{E}(\|x(t) - x_r(t)\|^2) &< \frac{1}{\underline{\alpha}}\Big(\frac{C_1 + C_2}{2\lambda} + \mathbb{E}\big(V(x_0, x_0^*)\big)e^{-2\lambda t} + \underline{\alpha}\rho_a^2\Big) \\ &\leq \frac{1}{\underline{\alpha}}\Big(\frac{C_1 + C_2}{2\lambda} + \bar{\alpha}\mathbb{E}\big(\|x_0^* - x_0\|^2\big)e^{-2\lambda t} + \underline{\alpha}\rho_a^2\Big), \end{aligned} \tag{B.22}$$

where the last line uses the property of IES Lyapunov function in Definition 4.4. From Equation (B.22), the contraction in the 2-Wasserstein distance can be shown just by taking the infimum over all couplings of ν_0 and ν_{r_0}, [14]. Define the shorthand $r(e) = \|m - n\|^2$ for $e = (m, n)$, and let $Z_t = (x(t), x_r(t))$. Let $P : [0, T] \times \mathbb{R}^{2n} \times \mathcal{B}(\mathbb{R}^{2n}) \to \mathbb{R}^+$ denote the transition function of the Markov process Z_t, and recall that $P(t, e, B) = P(Z_t \in B | Z_0 = e)$ almost surely. Now suppose π_0^* is an optimal coupling between ν_0 and ν_{r_0}. Thus, we have:

$$W_2^2(\nu_0, \nu_{r_0}) = \int r(e) d\pi_0^*(e).$$

Finally, let $\mathbb{E}_{\pi_0^*}$ denote the expectation with respect to the product measure formed from the measure π_0^* on

Z_0 and the independent Wiener measure describing Z_t. Then, we have:

$$\begin{aligned}
\mathbb{E}_{\pi_0^*}\big[r(Z_t)\big] &= \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} r(e) dP(t, e_0, B) d\pi_0^*(e_0) \\
&= \int_{\mathbb{R}^{2n}} \mathbb{E}[r(Z_t)|Z_0 = e_0] d\pi_0^*(e_0) \\
&\leq \int_{\mathbb{R}^{2n}} \frac{1}{\underline{\alpha}} \Big(\frac{C_1 + C_2}{2\lambda} + \bar{\alpha}\mathbb{E}\big(\|x_0^* - x_0\|^2\big) e^{-2\lambda t} + \underline{\alpha}\rho_a^2 \Big) d\pi_0^*(x_0^*, x_0) \\
&= \frac{1}{\underline{\alpha}} \Big(\frac{C_1 + C_2}{2\lambda} + \underline{\alpha}\rho_a^2 + \bar{\alpha} W_2^2(\nu_0, \nu_{r_0}) e^{-2\lambda t} \Big),
\end{aligned}$$

where the inequality in the third line uses the result from Equation (5.8). Referring to [14], define another measure $\pi_t(B) = \int P(t, e_0, B) d\pi_0^*(e_0)$. Thus, we have:

$$\mathbb{E}_{\pi_0^*}\big[r(Z_t)\big] = \int r(e) d\pi_t(e).$$

Recall the definition of 2-Wasserstein distance:

$$W_2^2(\nu_t, \nu_{r_t}) = \inf \int r(e) \pi_t(e).$$

Combining together, we finally have:

$$\begin{aligned}
W_2^2(\nu_t, \nu_{r_t}) &= \inf \int r(e) \pi_t(e) \\
&\leq \int r(e) \pi_t(e) \\
&\leq \frac{1}{\underline{\alpha}} \Big(\frac{C_1 + C_2}{2\lambda} + \underline{\alpha}\rho_a^2 + \bar{\alpha} W_2^2(\nu_0, \nu_{r_0}) e^{-2\lambda t} \Big).
\end{aligned}$$

Taking the square root on both sides, we have:

$$\begin{aligned}
W_2(\nu_t, \nu_{r_t}) &< \sqrt{\frac{1}{\underline{\alpha}}} \sqrt{\frac{C_1 + C_2}{2\lambda} + \underline{\alpha}\rho_a^2 + \bar{\alpha} W_2^2(\nu_0, \nu_{r_0}) e^{-2\lambda t}} \\
&\leq \sqrt{\frac{1}{\underline{\alpha}}} \big(\sqrt{\frac{C_1 + C_2}{2\lambda}} + \sqrt{\underline{\alpha}}\rho_a + \sqrt{\bar{\alpha}} W_2(\nu_0, \nu_{r_0}) e^{-\lambda t} \big).
\end{aligned}$$

□